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The Historical Development of Abelian Functions up to the time of Riemann. By HARRIS HANCOCK.

[Ordered by the General Committee to be printed *in extenso* among the Reports.]

(1) IN 1846, R. Leslie Ellis[1] presented to the British Association a 'Report on the Recent Progress of Analysis (Theory of the Comparison of Transcendents).' At the beginning of this memoir he says: 'The province of analysis, to which the theory of elliptic functions belongs, has within the last twenty years assumed a new aspect; in no subject, I think, has our knowledge advanced so far beyond the limits to which it was not long since confined.' 'This circumstance,' he continues, 'would give a particular interest to a history of the recent progress of the subject, even did it now appear to have reached its full development. But on the contrary, there is now more hope of further progress than at the commencement of the period of which I have been speaking.'

These statements appear more emphatic when we consider that after the lapse of fifty years, since the publication of Ellis's report to the present time, the same remarks are literally true, and when at the end of this period we find that there is more hope for the future progress of analysis, the theory of functions, than there has ever been before.

So great has been the growth of this science, extending on the one hand, and with a broadening influence, far into the realms of almost every branch of mathematical study, and on the other hand so comprehensive and varied in character is its application to physical problems, that the development of Ellis's work must be divided into many parts.

(2) The present report which the author has the honour of submitting to the Association is intended as a brief account of that part of the work already begun by Ellis which treats of the developments of the Abelian (including the hyperelliptic) functions. It is also found that the development of these functions has been so rapid and so extended that an adequate account of it would require much more space than can be given here. The author has consequently decided to make this statement for the period up to the time of Riemann. With Riemann, Weierstrass, Clebsch and Gordan, Cayley and others, the subject takes directions so essentially different that separate accounts along these different lines seem very desirable.

Much regarding the history of the general theory of functions may be found in Forsyth, 'Theory of Functions'; Harkness and Morley, 'A Treatise of the Theory of Functions'; Casorati, 'Teorica delle funzioni di variabili complesse'; Brill and Nöther, 'Die Entwicklung der algebraischen Functionen in älterer und neuerer Zeit' (see 'Jahresbericht der deutschen Mathematiker-Vereinigung,' 1894, bd. iii.). Fruitful sources for researches regarding the elliptic functions are Königsberger, 'Zur Geschichte der Theorie der elliptischen Transcendenten in den Jahren 1826–29,' Leipzig, 1879; short notices about the *first discovery of elliptic functions* are given by Gauss, 'Werke,' iii. p. 491; 'Correspondance mathématique entre Legendre et Jacobi' (Crelle's Journal, bd. lxxx. p. 205); and especially good is the account given by Enneper, 'Elliptische Functionen: Theorie und Geschichte,' Halle, 1890.

These works give more or less extended accounts of the subject under

[1] Ellis, *Report of the British Association for the Advancement of Science*, 1846, p. 34. We shall hereafter use the word 'Ellis' in referring to this paper.

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consideration ; other sources of information will be cited in their proper places.

(3) A good account (especially from the German standpoint) is given of the early development of the theory of functions by Brill and Nöther (*loc. cit.*). I shall here consider very briefly only such parts of the theory of elliptic functions that have a direct bearing upon this report, omitting as far as possible what has already been given by Ellis.

(4) The contributions towards the advancement of the elliptic functions by Tschirnhaus (1683–1700), the Bernoullis (1690–1730), Fagnano ('Produzioni Matematiche,' Pesaro, 1750) are discussed by Enneper ('Elliptische Functionen').

Two works which must have exercised great influence upon subsequent writers are Maclaurin, 'A Treatise on Fluxions,' Edinb. 1742, and d'Alembert, 'Recherches sur le calcul intégral' ('Histoire de l'Acad. de Berlin,' 1746, pp. 182–224).

(5) Euler extended and systematised the work that Fagnano had begun. It was known that the expressions for $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, etc., gave a means of adding or subtracting the arcs of circles, and that between the limits of two integrals that express lengths of arc of a lemniscate an algebraical relation exists, so that the arc of a lemniscate, although a transcendent of higher order, may be doubled or halved just as the arc of a circle by means of geometric construction.

It was natural to inquire if the ellipse, hyperbola, etc., did not have similar properties ; investigating such questions, Euler made the remarkable discovery of the addition-theorem of elliptic integrals (*cf.* 'Nov. Comm.' Petrop. vi. pp. 58–84, 1761 ; vii. p. 3 ; vii. p. 83).

Euler showed that if

$$\int_0^x \frac{d\xi}{\sqrt{\phi(\xi)}}+\int_0^y \frac{d\xi}{\sqrt{\phi(\xi)}}=\int_0^a \frac{d\xi}{\sqrt{\phi(\xi)}},$$

where $\phi(\xi)$ is a rational integral function of the fourth degree in ξ, there exists between the upper limits x, y, and a of the integrals an algebraic relation which is the addition-theorem of the arcs of an ellipse and is the algebraic solution of the differential equation[1]

$$\frac{d\xi}{\sqrt{\phi(\xi)}}+\frac{d\eta}{\sqrt{\varphi(\eta)}}=0.$$

Euler stated that the above results were obtained, not by any regular method, but *potius tentando, vel divinando*, and suggested that mathematicians seek a direct proof. The numerous discoveries of Euler are systematised in his work, 'Institutiones calculi integralis.'

The fourth volume (p. 446) contains an extension of the addition-theorem to integrals of the second and third kinds, as they were subsequently classified and named by Legendre.

In each case geometrical application of the formulæ are made for the comparison of elliptic arcs.

(6) The addition-theorem for elliptic integrals gave a similar meaning in higher analysis to the elliptic functions as the cyclometric and logarithmic functions had had for a long time. See Enneper ('Ellipt. Funct.,' p. 541 *et seq.*) regarding the position occupied by Euler in the development of the elliptic functions, and for a statement regarding Legendre's work in

[1] Euler, *Nov. Comm.* vol. x. pp. 3–50.

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this branch of mathematics *confer* Dirichlet's 'Gedächtnissrede auf Jacobi' (Jacobi's 'Werke,' vol. i. p. 9).

(7) The suggestion made by Euler that one should find a direct method of integrating the differential equation proposed by him (art. 5) was carried out by Lagrange, who by direct methods integrated this equation, and in a manner which elicited the great admiration of Euler. (See 'Miscell. Taurin.,' iv. 1768; or Serret's 'Œuvres de Lagrange,' vol. ii. p. 533.)

(8) The consideration of relations between integrals that have different moduli gave rise to a theorem due to Landen (and proved somewhat differently by Lagrange), in accordance with which an elliptic integral may be transformed into another integral of the same kind by means of algebraic transformations. Landen ('Phil. Trans.,' 1775, p. 285; or 'Mathematical Memoirs,' by John Landen, London, vol. i. 1780, p. 33) proves that in general the hyperbola may be rectified by means of two ellipses, with the addition of an algebraic quantity.[1]

The germ of the general theory of transformation is contained in this theorem, as has been observed by Legendre.[2]

By means of algebraic tranformations Landen was able to reduce elliptic integrals of the first kind into forms that had the same modulus, and showed that an elliptic integral of the first kind could be transformed into an elliptic integral of the first kind with smaller modulus, or into an integral of the first kind with smaller amplitude and greater modulus.

Lagrange[3] showed that the integration of any irrational function which contains the square root of a function ϕ may be made to depend upon the integration of a function of the form $\frac{P(x)}{\sqrt{\phi(x)}}$ where P is rational; and that if ϕ is not higher than the fourth degree in x, the integration may be reduced to that of

$$\frac{N dx}{\sqrt{(1 \pm p^2x^2)(1 \pm q^2x^2)}},$$

N denoting a rational function in x^2, and p and q constants. If the elliptic integral be reduced to this form, Lagrange showed by the introduction of a new variable that this integral may be transformed into another of similar form, but in which p and q become two new quantities p' and q', and that if p is greater than q, p' becomes greater than p and q' less than q. By the repetition of this process the factor corresponding to $1 \pm q^2x^2$ may be made as near unity as we desire, and consequently the integral may be expressed by a circular arc or logarithm; if, however, the transformations are made in the other direction, the functions corresponding to $1 \pm p^2x^2$ and $1 \pm q^2x^2$ become as near equal as we wish, and thus the elliptical integral reduces to a lower transcendent.[4]

Legendre investigated the general integral given above,

$$\int \frac{P dx}{\sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4}},$$

[1] An interesting geometric construction of this transformation is found in a letter of Jacobi to Hermite (Jacobi's *Werke*, bd. ii. p. 118). See also a geometric proof by MacCullagh (*Trans. of the Royal Irish Academy*, vol. xvi. p. 76).

[2] See Ellis, p. 37.

[3] *Mémoire de l'Acad. de Sc.*, 1784–85; *Œuvres* ii. p. 253.

[4] Ellis, p. 44. Casorati, *Teorica delle funzioni*, &c., p. 6.

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and showed that it was always possible to reduce it to one or the other of three forms essentially different.

We may mention, in passing, as being among the early English contributions to the subject memoirs by Brinkley ('Dublin Trans.' ix. p. 145, 1803) and Wallace ('Edinb. Trans.' v. p. 253). A criticism of Talbot's 'Researches on the Integral Calculus' ('Phil. Trans.' 1836, p. 177, and 1838, p. 1) is given by Ellis, p. 41.

(9) The theory of the elliptic functions, as Abel and Jacobi [1] found it in 1827, offered many highly enigmatical phenomena, which could not be explained by the principles that were at that time in vogue. For example, the degree of the equation which is found by means of Euler's theorem, and upon whose solution depends the division of the elliptic integral, was not, as in the analogous question of the division of the circle, equal to the number of the parts, but to the square of this number. It was easy to see the meaning of the real roots, whose number agrees with the number we have in the division of the circle; however, the number of imaginary roots must have seemed without explanation (Dirichlet, 'Gedächtnissrede,' p. 9).

We shall next consider the *inverse functions* of the integrals which we have been treating. With Jacobi [2] we begin with the simple algebraic integral

$$u=\int_0^x \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$$

In this expression we may either consider u as a function of the upper limit x, or inversely, the upper limit x as a function of u. In the first case, when $u=\sin^{-1}x$, it is not possible to express u in the form of a power series which is convergent for every value of x; and for a given value of x, u is not determinate, but has an infinite number of values, differing by multiples of 2π. But when we regard the upper limit x as a function of u, and write $x=\sin u$, then x may be expressed as a series which is convergent for all values, real and imaginary, of u; and when u is given a definite value, then x also has a definite value, and x considered as a function of u enjoys all the properties of a rational function.

The next more general algebraic integral is the elliptic integral

$$u=\int_0^x \frac{dx}{\sqrt{(1-x^2)\ (1-\kappa^2x^2)}}=\Pi(x).$$

As above, $u=\Pi(x)$ cannot be expressed by a series that is always convergent; and for a given value of x the variable u has not a definite value, but a double infinity of values, differing by multiples of the periods of elliptic functions (see next article).

The innate property of this integral could not be recognised if we considered the transcendent x alone; but we have to regard the upper

[1] Their first writings on this subject are: Abel, Crelle, bd. ii. September 1827; Jacobi, two letters to Schumacher dated June 13 and August 2, 1827, in the *Astronomische Nachrichten*, No. 123, vol. vi.

[2] Jacobi, *Considerationes generales de transcendentibus Abelianis* (*Werke*, bd. ii. p. 8).

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limit x as a function of u, and with Legendre we write $x=\sin\phi$, so that the integral above becomes

$$u=\int_0^{\phi}\frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}.$$

We consider ϕ as a function of u, and write $\phi=$amplitude of $u=\mathrm{am}(u)$, so that $x=\sin\mathrm{am}(u)=\mathrm{sn}\,u$.

The function $x=\mathrm{sn}\,u$ enjoys all the properties of a rational *fractional* function, and, as is shown later in connection with the θ-function, the numerator and denominator of this fraction may be developed in rapidly convergent series for all real or imaginary values of u. Hence the elliptic function $x=\mathrm{sn}\,u$ has one, and only one, definite value, corresponding to a given value of u.

(10) *Periods of the inverse functions.*—Abel and Jacobi recognised that the elliptic functions have at the same time the nature of circular functions and of exponential functions in that they are periodic for both real and imaginary values of the arguments. They saw that the function $x=\mathrm{sn}\,u$, for example, remained unaltered when u is changed into $u+4\mathrm{K}$ or into $u+2\mathrm{K}'\sqrt{-1}$, where K and K′ are definite constants.

Jacobi often repeated that the introduction of the imaginary was a complete solution of all the enigmas that had previously beset this subject.[1]

The introduction of the imaginary and the necessity of treating the limit as a function of the integral were two great advances made by Jacobi and Abel.

(11) Abel's investigations took different directions from those of Jacobi. Abel devoted himself to problems that have to do with the multiplication and division of elliptic integrals, their double periodicity, and their definition by infinite products. By the help of the principle of double periodicity he penetrated deeply into the nature of the roots of the equation upon which the division depends, and made the unexpected discovery that the general division of the elliptic integral with arbitrary limit may be performed algebraically (*i.e.* through the extraction of roots) as soon as the special division of the so-called *complete* integrals is presupposed performed.

The simplest case of this special division is for the modulus to which the lemniscate corresponds; and Abel shows that the division of the entire lemniscate is completely analogous to that of the circle, and may be performed by geometric construction in the same cases as the circle admits. The solution of the circle had been solved some twenty-five years before by Gauss. The admirers of Gauss with Dirichlet, from whom the above extracts have been made (*loc. cit.* p. 11), contend, from certain remarks (among others) made by Gauss in connection with the division of the circle and the lemniscate, that the principle of double periodicity was also known to Gauss. Some persons might, however, insist that Gauss too was beset by some of the enigmas above referred to, and that it was more likely that Gauss omitted to mention these dilemmas than to keep silent about the remarkable doubly periodic property of the functions that are connected with the lemniscate. In this connection Enneper ('Elliptische Functionen,' p. 7) says that it is to be regretted that Gauss did not communicate his remarkable discoveries to his contemporaries and invite their co-operation.

[1] See Dirichlet, *Gedächtnissrede auf Jacobi* (Jacobi's *Werke*, i. 10).

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Another important discovery is due to Abel's investigations: when the multiplier became infinitely large in the formulæ through which he represented the elliptic functions of a multiple argument by means of functions of the simple argument, he obtained remarkable expressions for the elliptic functions in form of infinite series that are expressed as quotients of infinite products.[1]

Jacobi, contemporaneously with Abel, was occupied in another part of the theory of elliptic functions, and with equally as great success. A fortunate induction of considering the transformation and the multiplication from a common point of view, and the last as a special case of the first, led him to the conjecture that rational functions of any degree may be used to transform an elliptic integral into an integral of the same form. This conjecture was at once confirmed, since the number of constants which may be arbitrarily disposed of for any degree is sufficient to satisfy all conditions in order that the form of the transformed integral may agree with the original (*cf.* Dirichlet, *loc. cit.* p. 12).

Jacobi also showed how the elliptic functions may be expressed in the form of infinite products, which may be represented by trigonometric series, and he further used the infinite series to express the square and product of these functions. These results, with the general theory of transformation, are systematised in the 'Fundamenta Nova,' Königsberg, 1827; Jacobi's 'Werke,' bd. i. p. 49; further developments in this direction are mentioned by Enneper ('Ellip. Funct.' p. 74 *et seq.*). A report of this work is given by Ellis (pp. 49–59). See also a paper by Poisson (Crelle, bd. x. p. 342).

(12). *Statement of Abel's Theorem*—We write, as above,

$$\text{(I.)} \qquad \int_0^x \frac{dx}{\sqrt{X}} = \Pi(x).$$

If in this expression $X = 1 - x^2$, and if there exists a given algebraic relation $x_1^2 + x_2^2 = 1$, then,

$$(1) \quad \Pi(x_1) + \Pi(x_2) = \text{Constant};$$

$$i.e. \text{ if } \sin^2\phi + \sin^2\psi = 1 \text{ then } \phi + \psi = \frac{\pi}{2}.$$

Further, if $X = (1-x^2)(1-\kappa^2x^2)$, and if we have given the algebraic relation [2]

$$4(1-x_1^2)(1-x_2^2)(1-x_3^2) = (2 - x_1^2 - x_2^2 + x_3^2 + \kappa^2 x_1^2 x_2^2 x_3^2)^2,$$

then is

$$(2) \quad \Pi(x_1) + \Pi(x_2) + \Pi(x_3) = 0.$$

From these two examples it is seen that, although in general we cannot integrate (I.) by means of algebraic or logarithmic functions, nevertheless, we have expressions (1) and (2) for the sums of such integrals, provided the variables that occur in these integrals are connected by algebraic relations.

By Euler's theorem, any number of elliptic integrals of the first kind may

[1] See in this connection Cayley, *Liouv. Journ.* x. p. 385; also numerous papers in his collected works:—Heine (Crelle, bd. xxxiv. p. 122); Eisenstein (Crelle, bd. xxxv. p. 153; Liouville (*Liouv. Journ.* t. ii. p. 433); Lipschitz (*Acta Math.* bd. iv. p. 193); Biermann (*Theorie der analytischen Functionen*, p. 323), &c.

[2] See a paper by Boole, 'On the comparison of transcendents,' *Phil. Trans.* 1857, p. 750; and also Rowe, 'Memoir on Abel's theorem,' *Phil. Trans.* Pt. III. 1881.

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be expressed by one such integral, where the upper limit of this integral is a rational function of the upper limits of the other integrals. Similar results are found for the elliptic integrals of the second and of the third kinds. For those of the second kind there enters, in addition, an algebraic function, and for those of the third kind a logarithmic algebraic function.

Abel considered the integrals of any algebraic functions, and established a theorem for the transcendents that arise from the integrals of these functions, which has for them the same meaning as Euler's theorem has for the elliptic transcendents.

The question proposed by Abel is: Suppose X in formula (I.) above is any algebraic function of x, then is it possible, taking different variables, to establish algebraic (or logarithmic) relations between integrals of the form

$$\Pi(x)=\int_0^x \frac{dx}{\sqrt{X}},$$

when the variables are connected by requisite algebraic equations; that is, can algebraic (or logarithmic) relations be found among

$$\Pi(x_1),\ \Pi(x_2),\ \ldots\ \Pi(x_n),$$

when $x_1, x_2, \ldots x_n$ are connected by algebraic equations? If such is the case, the question next arises: How many algebraic equations are necessary, and do these equations depend upon the nature of the function X?

Abel, in his celebrated paper, 'Mémoire sur une Propriété Générale d'une Classe Très-Étendue de Fonctions Transcendantes,' 'Œuvres Complètes,' t. i. p. 145 (Sylow and Lie), considered the question in a still more general form, and found that all those functions whose derivatives may be expressed through algebraic equations, in which the coefficients are rational functions of one and the same variable possess properties that are analogous to those of the elliptic functions stated above.

The results of these investigations are expressed in the following theorem, known as Abel's theorem: *If we have several functions whose derivatives may be (expressed as) the roots of one and the same algebraic equation, and all the coefficients in this algebraic equation are rational functions of one and the same variable, then it is always possible to express the sum of any number of functions which are like the first functions by means of an algebraic (and logarithmic) function, provided a certain number of algebraic relations can be established between the variables of the function in question.*[1]

The number of these relations does not depend upon the number of the

[1] Such functions are $\frac{dw_1}{dx} = \mathrm{R}\ (y_1,x_1)$, $\frac{dw_2}{dx} = \mathrm{R}\ (y_2,x_2)$, $\ldots$ $\frac{dw_n}{dx} = \mathrm{R}\ (y_n,x_n)$, R denoting a rational function, where $x_i (i = 1, 2, \ldots n)$ are the points of intersection of two curves $\chi(x,y) = 0$ and $\theta\ (x,y) = 0$, and y_i are the corresponding values of y that are obtained from these two equations.

Now every symmetric function of the solutions common to $\chi\ (x, y) = 0$, and $\vartheta(x,y) = 0$ is a rational function of the coefficients of these two equations.

Hence $\sum_{\kappa=1}^{\kappa=n} \int^{w_\kappa y_\kappa} \mathrm{R}(x,y)dx$ is an one-valued function of the coefficients of $\chi(x, y) = 0$,

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functions, but only upon the nature of the particular functions that are considered.

The same theorem is still true when we suppose the functions multiplied by any rational number positive or negative.

We may therefore deduce the following theorem: We are always able to express the sum of a given number of functions, which are multiplied each by a rational number, and in which the variables are arbitrary, by a similar sum of functions, whose number is determinate, and in which the variables are algebraic functions of the given functions.

As a further consequence is the theorem that the sum of any number of integrals of the form considered may be expressed by the sum of a definite number of such integrals with (perhaps) the addition of a determinate algebraic (and logarithmic) expression, in which the variables are algebraic functions of the variables of the first integrals.

We therefore have the following result: Although in general we cannot integrate an algebraic function by means of algebraic or logarithmic functions, we may, however, obtain for the sum of a certain number of such transcendental integrals an expression which is composed of algebraic (and logarithmic) functions.

Abel considered further the smallest number p of integrals through which the sum of any number of other integrals may be expressed. This is the well-known number which denotes the *class* (*Classen-zahl*) of the connectivity of Riemann's surface $\chi(x, y)$, upon which y is an one-valued function of x, and Clebsch's *deficiency* of the algebraic curve $\chi(x, y) = 0$. (See Cayley's 'Addition to Mr. Rowe's Memoir,' *loc. cit.* p. 752.)

Jacobi ('Werke,' bd. i. p. 379) writes: 'To this theorem we prefer to give as the most beautiful monument of [Abel's] extraordinary intellect, the name *Abel's theorem*, since it bears the entire stamp of his depth of thought. We consider it the greatest mathematical discovery of our time, as it in a simple form, and without the apparatus of the calculus, gives utterance to the deepest mathematical thought.'

Legendre calls the theorem a *monumentum ære perennius* (letter to Jacobi, Jacobi's 'Werke,' bd. i. p. 376).

The theorem is contained in a paper written in the year 1825, but not published until after Abel's death: 'Sur la comparaison des fonctions transcendantes' ('Œuvres,' t. ii. p. 55). The theorem so stated in this paper is: The sum of any number of functions which have an algebraic differential may be expressed through a definite number of such functions. It is developed in the large memoir above mentioned, 'Mémoire sur une propriété, etc.,' which was presented to the French Academy, October 1826, and not published until 1841 in the 'Mémoires des Savants Étrangers,' t. vii.

Legendre in the third supplement of the 'Traité des fonctions elliptiques,' p. 191, gives to the transcendent

$$\int \frac{f(x)\,dx}{\sqrt{X}},$$

and $\theta(x, y) = 0$, this one-valued function being by Abel's theorem an algebraic (and logarithmic) function.

The points (x_i, y_i) $(i = 1, 2, \ldots n)$ are not independent of each other, but as soon as a certain number of them is given, the remaining p (say) are of themselves determined, being the roots of an algebraic equation of the pth degree, whose coefficients are rational functions of those points that are given, so that between these coefficients there exist p algebraic relations.

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where $f(x)$ is a rational function of x and X a function of greater degree than the fourth in x, the name ultra-elliptic; when X is of the fifth or sixth degree in x, it is said to be of the *first order*;[1] when X is of the seventh or eight degree in x, of the *second order*, etc. In each order three different *kinds* of integrals are to be distinguished, which are entirely analogous to those 'which the nature of things has introduced into the theory of elliptic functions.'

Jacobi ('Werke,' bd. i. 1832, p. 376) wished to call these the *Abelian transcendents*, on account of the following works of Abel that had at that time appeared :—

'Remarques sur quelques propriétés générales d'une certaine sorte de fonctions transcendantes' (Crelle, bd. iii. 1828, p. 313).

'Démonstration d'une propriété générale d'une certaine classe de fonctions transcendantes' (Crelle, bd. iv. p. 200).

These papers treat of the more special functions in which y is connected with x by the relation $y^2=X$, where X has the same meaning as above; the term *hyperelliptic* is usually applied to such functions, *Abelian* being used *in general* to designate transcendents in which y is defined as any function of x through the algebraic equation $\chi(x,y)=0$.

(13) *Abel's theorem.*—A brief account of some of the fundamental statements of the preceding article is given here. The mode of procedure is nearer that of Riemann, Fuchs, and later writers than that of the original memoir. Some of the results as derived by Abel are given later.

The algebraic equation

$$\chi(x,y)=p_0+p_1y+p_2y^2+\ldots+p_{n-1}y^{n-1}+y^n=\chi(y)=0,$$

in which all the coefficients p are rational integral functions of x, and the integral function

$$\theta(x,y)=q_0+q_1y+q_2y^2+\ldots+q_{n-1}y^{n-1}=\theta(y)=0,$$

where the q's are likewise integral functions of x, when considered geometrically, represent two curves, which intersect in a certain number of points. In the coefficients of these two equations may appear quantities, $u, v, w, \ldots$, quantities quite indeterminate, upon which the coefficients depend.

The co-ordinates of the intersection of the two curves are functions of $u, v, w, \ldots$, so that we may write the two curves in the form

$$\left.\begin{aligned}\chi(x,y\,;\,u,v,w,\ldots)&=0,\\ \theta(x,y\,;\,u,v,w,\ldots)&=0.\end{aligned}\right\}$$

Let the points of intersection of the two curves be

$$x_1,y_1\,;\ x_2,y_2\,;\ x_3,y_3\,;\ \ldots x_\mu,y_\mu\,;$$

and when particular values $u^0,v^0,w^0,\ldots$ are given to $u,v,w,\ldots$, let the corresponding points of intersection be

$$x_1^0,y_1^0\,;\ x_2^0,y_2^0\,;\ x_3^0,y_3^0\ \ldots\ x_\mu^0,y_\mu^0.$$

[1] Legendre uses the word class. We may remark here that, when he divides integrals of the form $\int\frac{x^n dx}{\sqrt{\phi(x)}}$ into the three different *kinds*, he must first assume that n is less than $\frac{\lambda}{2}-1$ or $\frac{\lambda-1}{2}$, where λ is the degree of $\phi(x)$. See Richelot (Crelle, bd. xii. p. 185), where different forms of the integrals corresponding to the different kinds are considered.

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Let R (x,y) be any rational function of x and y, and form the sum

$$\sum_{\kappa=1}^{\kappa=p}\int_{x_\kappa^0,y_\kappa^0}^{x_\kappa,y_\kappa} \mathrm{R}(x_\kappa,y_\kappa)dx_\kappa.$$

For the present discussion it is in every respect sufficient to consider only one parameter u, and to specify the function $\theta(x,y)=0$, which we do by writing

$$\theta(x,y)=\phi(x,y)-u\psi(x,y)=0, \quad . \quad . \quad (1)$$

$$\text{or } u=\frac{\phi(x,y)}{\psi(x,y)}.$$

The rational function $\frac{\phi(x,y)}{\psi(x,y)}$ takes the value u_0 as often as it does any other value u.

Writing

$$\text{(I.)} \qquad w_\kappa=\int_{x_\kappa^0,y_\kappa^0}^{x_\kappa,y_\kappa} \mathrm{R}(x_\kappa,y_\kappa)dx_\kappa,$$

we have

$$\text{(II.)} \qquad \sum_{\kappa=1}^{\kappa=p} w_\kappa=\sum_{\kappa=1}^{\kappa=p}\int_{x_\kappa^0,y_\kappa^0}^{x_\kappa,y_\kappa} \mathrm{R}(x,y_\kappa)dx_\kappa.$$

In the expression (II.) we shall study w as a function of u.

From (I.)

$$\frac{dw_\kappa}{du}=\frac{dw_\kappa}{dx_\kappa}\frac{dx_\kappa}{du}=\mathrm{R}(x_\kappa,y_\kappa)\frac{dx_\kappa}{du} \quad . \quad . \quad . \quad . \quad (2)$$

Differentiate (1) with regard to u, and we have

$$\psi(x_\kappa,y)+u\left[\left(\frac{\partial\psi}{\partial x}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}+\left(\frac{\partial\psi}{\partial y}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}\frac{dy_\kappa}{dx_\kappa}\right]\frac{dx_\kappa}{du}$$

$$-\left[\left(\frac{\partial\phi}{\partial x}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}+\left(\frac{\partial\phi}{\partial y}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}\frac{dy_\kappa}{dx_\kappa}\right]\frac{dx_\kappa}{du}=0 \quad . \quad (3)$$

Since $\left(\frac{\partial\chi}{\partial y}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}\frac{dy_\kappa}{dx_\kappa}+\left(\frac{\partial\chi}{\partial x}\right)_{\substack{x=x_\kappa\\y=y_\kappa}}=0$, we have, after substituting the value of $\frac{dy_\kappa}{dx_\kappa}$ from this equation in (3), a formula for $\frac{dx_\kappa}{du}$ which is rational in x_κ, y_κ and u. We may therefore write $\frac{dx_\kappa}{du}=\mathrm{S}(x_\kappa,y_\kappa,u)$, where S denotes a rational function.

Further, $\frac{dw_\kappa}{du}=\mathrm{R}(x_\kappa,y_\kappa)\,\mathrm{S}(x_\kappa,y_\kappa,u)=\mathrm{T}(x_\kappa,y_\kappa,u)$ where T is also a rational function.

Finally, $\frac{d}{du}\sum_{\kappa=1}^{\kappa=p} w=\sum_{\kappa=1}^{\kappa=p}\mathrm{T}(x_\kappa,y_\kappa,u)=r(u)$, say, where $r(u)$ is a rational function in u.

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Hence upon integrating

$$\text{(III.)} \qquad \sum_{\kappa=1}^{\kappa=\mu} w_\kappa = \sum_{\kappa=1}^{\kappa=\mu} \int_{x_\kappa^0, y_\kappa^0}^{x_\kappa, y_\kappa} \mathrm{R}(x_\kappa, y_\kappa) dx_\kappa = \int r(u)\, du.$$

We note that the number μ does not depend upon the function $\mathrm{R}(x,y)$.

(14) We thus have the sum of μ integrals expressed as the integral of a rational function of the parameter u. This integral depends upon the nature of the function $\mathrm{R}(x,y)$ being, first, a constant; secondly, the differential of a logarithmic function which is equivalent to a determinate algebraic function; thirdly, a logarithmic expression which likewise is determinate, when the integral $\int \mathrm{R}(x,y)dx$ are special integrals of the *first*, *second*, and *third* kinds respectively.

The discussion of these special integrals, the normal integrals of the first, second and third kinds, is found in Forsyth, 'Theory of Functions,' p. 443; Harkness and Morley, 'A Treatise on the Theory of Functions,' p. 435; Neumann, 'Theorie der Abel'schen Integrale,' p. 245, &c.

(15) Nöther [1] has proved that any algebraic curve may by means of birational transformations be transformed into another curve in which the highest singularities are double points with distinct tangents. We may therefore assume that the curve $\chi(x,y)=0$ has no higher singularities than these. Upon this hypothesis the most intricate integral that arises may be expressed linearly in terms of the normal integrals of the first, second and third kinds, with the addition, perhaps, of an algebraic expression.

Abel allowed the curve $\chi(x,y)=0$ to have any kind of singularity; and hence the expression for the algebraic and logarithmic functions that stand on the right of formula (III.) in art. 13 are necessarily very complicated. By making use of the methods mentioned above, this complexity is avoided, and the representation of the integral $\int r(u)du$ may be obtained in comparatively simple form.

(16) Denote the p linearly independent normal integrals of the first kind by $v_h(x,y)(h=1, 2, \ldots p)$; then, as in art. 13,

$$\sum_{\kappa=1}^{\kappa=p} \int_{x_\kappa^0 y_\kappa^0}^{x_\kappa, y_\kappa} dv_h(x,y)dx \equiv 0 \text{ (mod. } const.),$$

$$\text{or } \sum_{\kappa=1}^{\kappa=q} \int_{x_\kappa^0, y_\kappa^0}^{x_\kappa, y_\kappa} dv_h(x,y)dx + \sum_{\kappa=1}^{\kappa=p} \int_{x^0_{q+\kappa}, y^0_{q+\kappa}}^{x_{q+\kappa}, y_{q+\kappa}} dv_h(x,y)dx \equiv 0 \text{ (mod. const.)},$$

where $p+q=\mu$.

The points x_κ^0, y_κ^0 $(\kappa = 1, 2, \ldots q)$; $x^0_{q+\kappa}, y^0_{q+\kappa}$ $(\kappa = 1, 2, \ldots p)$ and the points x_κ, y_κ $(\kappa = 1, 2, \ldots q)$ may be chosen at pleasure; but the remaining p points $x_{q+\kappa}, y_{q+\kappa}$ $(\kappa = 1, 2, \ldots p)$ are no longer arbitrary, the x's being the roots of an algebraic equation of the pth degree

$$x^p + a_1 x^{p-1} + a_2 x^{p-2} + \ldots + a_p = 0,$$

[1] Nöther, *Math. Ann.* bd. ix. p. 17; see also Halphen, *Bulletin de la Soc. Math. de France*, t. iv. Dec. 1875, and t. iii. Feb. 1875; Bertini, *Rivista di Matem.* 1891, and *Math. Ann.* bd. xliv. p. 158; Poincaré, *Compt. Rend.* July 1893.

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where the constants $a_1, a_2, \ldots a_p$ are determinate rational functions of the other points,

$$x^0_\kappa, y^0_\kappa \ (\kappa = 1, 2 \ldots q) \ ; \ x^0_{q+\sigma}, y^0_{q+\sigma} \ (\kappa = 1, 2 \ldots p)$$
$$\text{and } x_\kappa, y_\kappa (\kappa = 1, 2 \ldots q).$$

The corresponding values of y are the remaining ordinates of the points of intersection of the curves $\chi(x,y) = 0$, and $\theta(x,y) = 0$.

(17) Abel, in the proof of his theorem, stated in art. 12 wrote :

$$\phi(x) = \int f(x,y)\, dx,$$

where $f(x,y)$ is any rational algebraic function of x and y. He then considered the sum of such integrals

$$\sum_{i=1}^{i=\mu} \phi(x_i),$$

where $x_i (i = 1, 2, \ldots \mu)$ are the points of intersection of the two curves $\chi(x,y) = 0$, $\theta(x,y) = 0$; that is, the roots of the equation $E(x) = 0$, which is obtained by eliminating y out of the two given equations.

Of these points of intersection some may be stationary, while the others are movable, the fixed points being independent of the parameters $u, v, w, \ldots$ (art. 13). Hence $E(x)$ may be composed of two factors $F_0(x)$ and $F(x)$, of which $F_0(x)$ does not depend upon $u, v, w, \ldots$

Abel wrote the subject of integration in the form

$$\frac{f_1(x,y)}{f_2(x,y)\, \chi'(y)},$$

where $f_1(x,y)$ and $f_2(x,y)$ are integral functions of x and y and $\chi'(y) = \frac{\partial \chi(x,y)}{\partial y}$.

He found that $\sum_{i=1}^{i=\mu} \phi(x_i) = \sum_{i=1}^{i=\mu} \frac{f_1(x_i, y_i)}{f_2(x_i, y_i)\, \chi'(y_i)} dx_i = v,$

where v may be a constant *plus* an algebraic function *plus* a logarithmic function. (A concise expression for v, due to Rowe, is given in art. 20.)

(18) After restricting the functions $f_1(x,y), f_2(x,y)$ and $F_0(x)$ in such a way that the logarithmic and algebraic functions of the expression above disappeared, Abel found that the function $f_1(x,y)$ contained a certain number of arbitrary constants, a number which depended only upon the nature of the curve $\chi(x,y) = 0$. This number he designated by $\gamma(= p$, of art. 13).

In the equation $\theta(x,y) = q_0 + q_1 y + q_2 y^2 + \ldots + q_{n-1} y^{n-1} = 0$, a certain number of the coefficients of x in the functions q are supposed indeterminate. Denote these by $a, a_1, a_2, \ldots$ We saw above that the upper limits $x_i (i = 1, 2, \ldots \mu)$ of the integrals in (I.) are the roots of the equation $E(x) = 0$, and may be expressed as functions of the independent quantities $a, a_1, a_2, \ldots$, of which there are, say, α.

Let these functions be :

$$x_1 = f_1(a, a_1, a_2, \ldots), x_2 = f_2(a, a_1, a_2, \ldots) \ldots x_\alpha = f_\alpha(a, a_1, a_2, \ldots).$$

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From these relations it is seen that as soon as a of the x's are given, the remaining $\mu - a$ may be determined in terms of the known ones. Abel showed how to effect this determination, and in general that $\mu - a = \gamma$.

In two special cases considered by Abel this number is less than γ. (See also Rowe, Memoir on Abel's Theorem, 'Phil. Trans.' 1881, p. 731.) Professor Cayley, in the 'Addition to Mr. Rowe's Memoir,' proved that γ was always equal to the deficiency of the curve $\chi(x,y) = 0$, whatever its singularities.

Professor H. F. Baker has recently proved the same theorem by means of graphic methods in the 'Cambr. Trans.' xv. Part IV.; see also 'Math. Ann.,' 45, p. 133.

As a special case Abel gave the equation $\chi(x,y) = 0$ the form $y^n + p_0 = 0$.

The form of the integrals whose sum is to be expressed as in formula (I.) is

$$\int \frac{f_3(x)dx}{f_2(x)y^m}.$$

For the hyperelliptic functions ($n = 2$), when p_0 is of the $2m - 1^{st}$ or $2m^{th}$ degree, Abel showed that $\mu - a = m - 1$.

(19) Mathematicians were much interested in the new functions which must be introduced in connection with the Abelian integrals. The Academy at Copenhagen wished to see these functions extended to all integrals of algebraic functions, which are included in Abel's theorem;[1] and in regard to this wish Jürgensen, Broch, Minding, Rosenhain wrote some very important memoirs. The value of these memoirs, however, on account of their less generality was much diminished when Abel's great paper was finally published in 1841.

Minding, in two short papers (Crelle, bd. ix. p. 295, 1833, and bd. xi. p. 233, 1834), showed how to represent the algebraic and logarithmic functions of Abel's theorem for the special cases in which the algebraic functions satisfy an equation of the third degree.

Jürgensen ('Sur la Sommation des Transcendantes à différentielles algébriques,' Crelle, bd. xix. p. 113), took, as the subject of integration, the quotient of two functions $P(x, z_i)$ and $Q(x, z_i)$, where P and Q are integral functions, and where z_i is a root of an equation that is similar to Abel's $\chi(x,y) = 0$.

After reducing $\frac{P(x, z_i)}{Q(x, z_i)}$ to a form $\frac{\lambda(x, z_i)}{\nu(x)}$, where λ and ν denote integral functions (see Liouville, 'Note sur la Détermination des intégrals dont la valeur est algébrique,' Crelle, bd. x. p. 347),[2] he considered a sum of integrals of the form $\int \frac{\lambda(x, z_i),}{\nu(x)}$ where the summation is taken over the μ roots of the resultant of two algebraic equations. This sum he expressed in the form of an algebraic and logarithmic function.

In a second paper (Crelle, bd. xxiii. p. 126) Jürgensen denoted by $X(x, y_i)$

[1] *Cf.* Jacobi, *Gesam. Werke*, bd. ii. p. 517. He does not mention Jürgensen.

[2] See also references cited in art. 11, and a paper by Liouville, *Sur l'intégration d'une classe de fonctions transcendantes*, bd. xiii. p. 93.

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a rational function of x and any one of the n roots y_i ($i=1, 2, \ldots n$) of the equation

$$y^n+p_1 y^{n-1}+p_2 y^{n-2}+ \ldots +p_{n-1}y+p_n=0,$$

the p's being integral functions of x; then X (see Liouville's paper mentioned above) may be given one or the other of the forms

$$X=\frac{f(x)}{\phi_\kappa(x)} \text{ or } X=f(x)\,\phi_\kappa(x),$$

where $f(x)$ is a rational function of x, and $\phi_\kappa(x)$ is an integral function of x and y_κ.

Two leading questions are considered: (1) To find the cases in which one can express $\int X dx$ by a finite number of algebraic and logarithmic operations (see art. 34); (2) To find the relations among the integrals

$$\int f(x_1)\phi_\kappa(x_1)dx_1, \quad \int f(x_2)\phi_\lambda(x_2)dx_2, \ \ldots$$

which correspond to variables $x_1, x_2, \ldots$ that depend upon one another, and upon the different roots $y_\kappa, y_\lambda, \ldots$.

Broch, 'Mémoire sur les fonctions de la forme,'

$$\int x^{s-\nu p-1} f(x^p) \left(R(x^p)\right)^{\pm \frac{s}{rp}} dx,'$$

(Crelle, bd. xxiii. p. 148, 1841), developed rules for the summation of the transcendents mentioned in the title, where $f(x^p)$ is a rational function of x^p, γ an integer which is divisible by $\frac{s-1}{\gamma}$, r and p are integers, and s an integer less than $r.\ p$.

These are analagous to the investigations of Abel on the hyperelliptic functions which had already been published.

In a previous memoir (Crelle, bd. xx. p. 178), Broch had discussed the special case where $p=1$ and $s=1$. The basis of this paper is Abel's memoir, 'Démonstration d'une propriété générale,' &c. Broch also sought the minimal number of integrals (Abel's γ), through which a sum of integrals could be represented.

Minding divided his paper, 'Propositiones quædam de integralibus functionum algebraicarum,' &c. (Crelle, bd. xxiii. p. 255), into three heads. He first gives an expression for a sum of integrals of the form

$$\int \frac{\phi_o(x_i)\ F(x_i, y_i p_o)\, dx_i}{\phi(x_i)},$$

when ϕ_o and F are integral functions, and

$$\phi(x)=(x-c_1)(x-c_2) \ \ldots \ (x-c_r),$$

the c's denoting constants. x_i and y_i are the common intersections of two curves

$$p_0 y^n+p_1 y^{n-1}+ \ \ldots \ +p_n=0\ ;\ q_1 y^{n-1}+q_2 y^{n-2}+ \ \ldots \ +q_n=0,$$

which correspond to Abel's $\chi(x,y)=0$ and $\theta(x,y)=0$.

Minding further allowed arbitrary variable parameters in his functions q, so that his results, as Brill and Nöther [1] remark, are only less general than those of Abel in that fixed points of intersection of the two curves are not considered.

[1] *Jahresber. der deutschen Mathematiker-Vereinigung*, bd. iii. p. 229.

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The contents of the second head are indicated by its title 'De numero minimo integralium ad quæ numerus datus eiusmodi integralium reduci potest.'

In the third head he makes application of the preceding theorems to the equation : $p_0y^n+p_n=0$. (See also art. 19.)

Ramus (Crelle, bd. xxiv. p. 69) derives a formula for the expression of the sum of μ integrals $\psi(x_1)+\psi(x_2)+ \ldots +\psi(x_\mu)$, where $\psi(x)$ has either the form $\int\frac{f(x)}{x-a}y^m(x)\,dx$ or $\int\frac{f(x)}{x-a}\frac{dx}{y^m(x)}$, where $f(x)$ is an integral function of x, m a positive integer, and $y^m(x)$ is one of the n roots $y_1(x)$, $y_2(x)$, . . $y_n(x)$ of the equation

$$0=p_0+p_1y+p_2y^2+ \ldots p_{n-1}y^{n-1}+y^n.$$

(Abel's $\chi(x,y)=0$.)

The variables $x_i(i=1, 2, \ldots \mu)$ are the points of intersection of this curve with a second algebraic curve.

Rosenhain (Crelle, bd. xxviii. p. 249, 1844) employed as fundamental equation

$$\phi(x,y)=p_0y^n+p_1y^{n-1}+ \ldots +p_n=0.$$

in the place of Abel's $\chi(x,y)=0$.

Proceeding in a manner very similar to that of Abel, he adopts in his summation-formula integrands of the form

$$\frac{Q_2y^{n-2}+Q_3y^{n-3}+ \ldots + Q^n}{\phi'(y)},$$

where the Q's are the rational functions of x.

In discussing the hyperelliptic case ($n=2$) he gives $\phi(x,y)$ the form $p_0y_2+p_1y+p_2$, and not the form usually adopted, $y^2=R(x)$, p_0, p_1, p_2, and R denoting rational functions of x.

He then seeks to prove in the general case that the number of arbitrary constants in an integral of the first kind is equal to the smallest number of integrals through which the sum of any number of such integrals is expressible. The article is continued (Crelle, bd. xxix. p. 1).

(20) Boole, in a paper, 'On the Comparison of Transcendents, with certain Applications to the Theory of Definite Integrals' ('Phil. Trans.' 1857, p. 745), contemplates the following objects :

First, the demonstration of a fundamental theorem for the summation of integrals, whose limits are determined by the roots of an algebraic equation. Secondly, the application of that theorem to the comparison of algebraic transcendents. Thirdly, the application of the same theorem in a new, and, as it is conceived, more remarkable line of investigation to the comparison of functional transcendents. In the introduction to this paper, Boole states : 'As presented in the writings of Abel and of those who immediately followed in his steps, the doctrine of the comparison of transcendents is repulsive from the complexity of the formulæ in which the general conclusions are embodied.' With the intention of simplifying these formulæ, Boole introduced a symbol differing in interpretation only by the addition of one element from the symbol used by Cauchy in the 'Calculus of Residues.'

This symbol, Θ, he defines as follows : 'If $\phi(x)f(x)$ be any function of x composed of two factors $\phi(x)$ and $f(x)$, whereof $\phi(x)$ is rational, let $\Theta[\phi(x)]f(x)$ denote the result obtained by successively developing the

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function in ascending powers of each distinct simple factor (of the form) $x-a$ in the denominator of $q(x)$, taking in each development the coefficient of $\frac{1}{x-a}$, adding together the coefficients thus obtained from the several developments, and subtracting from the result the coefficient of $1/x$ in the development of the same function in descending powers of x.'

The simplifications made by Boole are more than offset by the loss of generality which characterise his formulæ.

Rowe (Memoir on Abel's Theorem, 'Phil. Trans.' 1881, p. 713) endeavoured to simplify Abel's results, and at the same time to retain their generality. Using the same notation as in art. 18, and employing Boole's symbol, he derives Abel's theorem in the form :

$$\Sigma\int f(x,y)dx=\Sigma\int\frac{f_1(x,y)dx}{f_2(x,y)\chi'(y)}$$
$$=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]F_0(x)\,\Sigma\frac{f_1(x,y)}{\chi'(y)}\log\theta(y)+C,$$

where C is a constant.

Professor A. R. Forsyth, in a paper, 'Abel's Theorem and Abelian Functions' ('Phil. Trans.' 1883, p. 323) has obtained an expression for an integral that is more general than that occurring in Abel's theorem.

Professor Forsyth takes two given equations of degrees m and n between three variables, of which y and z are dependent, x being independent :

$$F_m(x,y,z)=0,\ F_n(x,y,z)=0.$$

The Jacobian (functional-determinant) of these two functions is denoted by $J\left(\frac{F_m, F_n}{y, z}\right)$. A quantity T is defined by the relation

$$T=\frac{\Psi(x,y,z)}{\Phi(x,y,z)},$$

where Ψ and Φ are rational algebraic functions of x, y, z. This quotient may in turn be expressed in the form $\frac{U}{f(x)}$, where U is an integral function of x, y, and z, and $f(x)$ a function of x alone.

The generalised Abel's theorem, as derived by Professor Forsyth, is

$$\Sigma\int\frac{U\,dx}{f(x)J\left(\frac{F_m, F_n}{y, z}\right)}=\Theta\left[\frac{1}{f(x)}\right]\Sigma\left\{\frac{U}{J\left(\frac{F_m, F_n}{y, z}\right)}\log F_p\right\}+C,$$

where the upper limits of the integrals on the left-hand side are the $m.n.p$ roots of the equation obtained by eliminating y and z between F_m and F_n, and an arbitrary rational algebraic function $F_p(x,y,z)=0$. On the right-hand side the summation extends over the $m.n$ roots y and z in terms of x of the equations $F_m=0$ and $F_n=0$. C is a constant.

The more general theorem Professor Forsyth enunciates as follows : Let $F_i(x_1, x_2, \ldots x_r)=0$ $(i=1, 2, \ldots r-1)$ be $r-1$ algebraic equations of degrees $m_1, m_2, \ldots m_{r-1}$ respectively, giving $x_2, x_3, \ldots x_r$ in terms of x_1 ; and let $F_r(x_2, x_3, \ldots x_r)$ be a function of these dependent variables, the coefficients of which are functions of x, containing any number of arbitrary constants. Form the eliminant E of all the F's, so that we

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shall obtain the set of roots x_1 by equating E to zero; and denote by U any algebraic function of $x_1, x_2, \ldots x_r$.

Then

$$\Sigma \int \frac{U}{f(x_1)} \frac{dx_1}{J\left(\frac{F_1, F_2, \ldots F_{r-1}}{x_2, x_3, \ldots x_r}\right)} = \Theta\left[\frac{1}{f(x_1)}\right] \Sigma \frac{U \log F_r}{J\left(\frac{F_1, F_2, \ldots F_{r-1}}{x_2, x_3, \ldots x_r}\right)} + A.$$

The summation on the right-hand side is taken over all the roots of E=0, which are assumed as the upper limits of the integrals; while on the right-hand side the summation is over all the roots $F_1=0$, $F_2=0$, . . . $F_{r-1}=0$, considered as $r-1$, simultaneous equations giving $x_2, x_3, \ldots x_r$ in terms of x_1.

In connection with this paper we note a paper by Professor Cayley, 'A Memoir on the Abelian and Theta Functions' ('American Journ. Math.' vol. v. p. 137, and vol. vii. p. 101).

The first chapter treats of Abel's theorem; the second, a proof of Abel's theorem. The connection between the lines of thought presented in this paper and those of Professor Forsyth are particularly interesting. In the further developments of Professor Cayley's paper, which is founded upon Clebsch and Gordan's 'Treatise,' some geometrical results are brought into prominence. The theory is illustrated by examples in regard to the cubic, the nodal quartic and the general quartic respectively.

The general case where the fixed curve is any curve whatever has been solved with great generality by Nöther,[1] 'Zur Reduction Algebraischer Differentialausdrücke auf der Normalform,' and 'Ueber die Algebraischen Differentialausdrücke' ('Sitzungsber. der Phys. Med. Soc. zu Erlangen,' Dec. 10, 1883, and Jan. 14, 1884). Other addition-theorems, especially for the hyperelliptic functions, are given in art. 32.

(21) *Periodic Functions of Several Variables.*—In art. 10 the periodic properties of functions of one variable were considered, and it has been seen that Abel's theorem embraces the integrals of all algebraic functions. Considering the inverse of these transcendental integrals, Jacobi discovered the existence of the *periodic functions of several variables*, and thus revealed the real significance and hitherto hidden properties of such functions.

Some of Jacobi's investigations[2] relative to hyperelliptic transcendents are next given, since they may be used to illustrate Abel's theorem for the more general integrals, and set forth the properties of the inverse functions that are comprised in this theorem.

If X denotes a rational integral function of the fourth degree in x, then by Euler's theorem (art. 5) transcendents of the form

$$\int_0^x \frac{dx}{\sqrt{X}} = \Pi(x)$$

enjoy the singular property that if

$$\Pi(x_1) + \Pi(x_2) = \Pi(a),$$

then a may be found algebraically in terms of x_1 and x_2. Owing to

[1] See also Nöther, *Math. Ann.* bd. ii. p. 314, bd. ix. p. 17; Brill and Nöther, *Math. Ann.* bd. vii. p. 269, and continuation in bd. vii.; Klein-Fricke, *Elliptische Modulfunctionen*, bd. i. 1890, p. 533; Baker, *Cambridge Phil. Trans.* xv.; *Math. Ann.* bd. xlv. p. 133, &c.

[2] Jacobi, *Considerationes Generales de Transcendentibus Abelianis*, Crelle, bd. ix. p. 394, 1832; *Werke*, bd. ii. p. 7.

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Abel's theorem, analogous properties exist for all such transcendents, in which the function X is any rational integral function of x. For, taking the next simplest case, let X be of the fifth or sixth degree in x, and write :

$$\int_0^x \frac{dx}{\sqrt{X}} = \Phi(x), \quad \int_0^x \frac{x\,dx}{\sqrt{X}} = \Phi_1(x);$$

and further write :

$$\Pi(x) = \int_0^x \frac{(A + A_1 x)dx}{\sqrt{X}},$$

or

$$\Pi(x) = A\Phi(x) + A_1\Phi_1(x),$$

where A and A_1 denote constants.

Then from Abel's theorem it follows that of the innumerable solutions of the equation

$$(1)\quad \Pi(x_1) + \Pi(x_2) + \Pi(x_3) = \Pi(a) + \Pi(b),$$

there is one algebraic solution ; that is, a and b may be algebraically determined in terms of x_1, x_2, x_3 from the two equations that are derived from (1) :

$$\Phi(x_1) + \Phi(x_2) + \Phi(x_3) = \Phi(a) + \Phi(b),$$
$$\Phi_1(x_1) + \Phi_1(x_2) + \Phi_1(x_3) = \Phi_1(a) + \Phi_1(b).$$

(22) In general, if we write :

$$\int \frac{(A + A_1 x + A_2 x^2 + \ldots + A_{m-2}x^{m-2})dx}{\sqrt{X}} = \Pi(x),$$

where $X = f(x)$ is a rational integral function of the $2m$th or $2m-1$th degree in x, it follows from Abel's theorem that, if m values $x_1, x_2, \ldots x_m$ of the variable x be given, through these m quantities it is possible to determine (art. 18) in an *algebraic* manner $m-1$ quantities $a_1, a_2, \ldots a_{m-1}$, which satisfy the transcendental relation :

$$\Pi(x_1) + \Pi(x_2) + \ldots + \Pi(x_m) = \Pi(a_1) + \Pi(a_2) + \ldots + \Pi(a_{m-1});$$

and Abel further showed that the quantities $a_1, a_2, \ldots a_{m-1}$ are the roots of an algebraic equation of the $m-1$th degree, and that each of the coefficients in this algebraic equation may be rationally expressed in terms of the quantities : $x_1, x_2, \ldots x_m$, and

$$\sqrt{X_1}, \sqrt{X_2}, \ldots \sqrt{X_m}, \text{ where } X_\lambda = f(x_\lambda)_{(\lambda=1,2\ldots m)}.$$

It also follows from Abel's theorem that, when any number whatever of values of x are given, the sum of the transcendents $\Pi(x)$ which belong to given values of x may be expressed through $m-1$ transcendents $\Pi(x)$ when the $m-1$ values of x in these transcendents are algebraically determined from the given values.

(23) We consider next the case where the sum of four transcendents are expressed as the sum of two, and where the arguments of these two transcendents depend algebraically upon the arguments of the first four. As above, we write :

$$\int_0^x \frac{dx}{\sqrt{X}} = \Phi(x) \text{ and } \int_0^x \frac{x\,dx}{\sqrt{X}} = \Phi_1(x).$$

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By Abel's theorem, when the two equations

$$(1)\begin{cases}\Phi(a)+\Phi(b)=\Phi(x_1)+\Phi(x_2)+\Phi(x_3)+\Phi(x_4),\\ \Phi_1(a)+\Phi_1(b)=\Phi_1(x_1)+\Phi_1(x_2)+\Phi_1(x_3)+\Phi_1(x_4)\end{cases}$$

are simultaneously given, a and b are algebraically determined in terms of the given quantities x_1,x_2,x_3,x_4.

Now write

$$(2)\begin{cases}\Phi(x_1)+\Phi(x_2)=u,\\ \Phi_1(x_1)+\Phi_1(x_2)=v;\end{cases}\qquad (2')\begin{cases}\Phi(x_3)+\Phi(x_4)=u',\\ \Phi_1(x_3)+\Phi_1(x_4)=v'.\end{cases}$$

Then from (1) it follows that

$$(3)\begin{cases}\Phi(a)+\Phi(b)=u+u',\\ \Phi_1(a)+\Phi_1(b)=v+v'.\end{cases}$$

If now in (2) we consider x_1 and x_2 as functions of u and v and write $x_1=\lambda(u,v),x_2=\lambda_1(u,v)$, and similarly in (2′) write $x_3=\lambda(u',v'),x_4=\lambda_1(u',v')$, then it follows from (3) that $a=\lambda(u+u',v+v')$, $b=\lambda_1(u+u',v+v')$.

Since a and b are algebraically expressible in x_1,x_2,x_3,x_4, it follows that $\lambda(u+u',v+v')$ and $\lambda_1(u+u',v+v')$ are algebraically expressible in terms of $\lambda(u,v,)$ $\lambda_1(u,v)$, $\lambda(u',v')$ and $\lambda_1(u',v')$.

The general theorem may be expressed as follows:

$$\text{Let}\int_0^x\frac{x^i dx}{\sqrt{X}}=\Phi_i(x),\ i=(0,1,\ldots m-2),$$

where $X=f(x)$ is a rational integral function of the $2m$th or $2m-1$th degree in x, then, if between the $m-1$ quantities $x_0,x_1,\ldots x_{m-2}$ and the quantities $u_0,u_1,\ldots u_{m-2}$ the following equations exist simultaneously,

$$(\text{I.})\quad u_i=\Phi_i(x_0)+\Phi_i(x_1)+\ldots+\Phi_i(x_{m-2}),\quad (i=0,1,\ldots m-2);$$

and if

$$(\text{II.})\quad x_i=\lambda_i(u_0,u_1,\ldots u_{m-2}),\quad (i=0,1,\ldots m-2);$$

with equations similar to (I.) and (II.) with accented u's and x's, then these functions enjoy the same property as do the trigonometric and elliptic functions, viz.:

The functions

$$\lambda_i(u_0+u_0',u_1+u_1',\ldots u_{m-2}+u'_{m-2})$$

may be algebraically expressed through the functions

$$\lambda_i(u_0,u_1,\ldots u_{m-2})\text{ and }\lambda_i(u'_0,u'_1,\ldots u'_{m-2}),\quad (i=0,1,\ldots m-2).$$

(24) *Integrals of differential equations.*—Euler's theorem sets forth the complete algebraic integral of a differential equation of the first order with two variables, which have been separated in such a way that

$$\frac{dx_1}{\sqrt{X_1}}+\frac{dx_2}{\sqrt{X_2}},$$

where X_1 and X_2 denote the same rational integral function of the fourth degree in x_1 and x_2 respectively; and Euler [1] showed that the algebraic integral was an equation of the second degree between the two quantities x_1+x_2 and $x_1.x_2$.

Abel's theorem sets forth *algebraically* $m-1$ complete integrals (integrals which involve $m-1$ arbitrary constants) of $m-1$ differential

[1] Euler, *Institutiones Calc. Int.* t. i. cap. vi. § 2.

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equations of the first order with m variables, in each of which the m variables are separated.

Taking the next simplest case, let $X=f(x)$ denote a rational integral function of the fifth or sixth degree. Then the two transcendental equations (see the preceding article)

$$\Phi(x_1)+\Phi(x_2)+\Phi(x_3)=\Phi(a)+\Phi(b),$$
$$\Phi_1(x_1)+\Phi_1(x_2)+\Phi_1(x_3)=\Phi_1(a)+\Phi_1(b),$$

owing to Abel's theorem, take the place of two algebraic equations between the five qualities x_1, x_2, x_3, a and b. Consider a and b as constants. Then, when we differentiate the two equations just written, the terms involving a and b drop out, so that these quantities appear as arbitrary constants in the transcendental equations, or in the algebraic equations which take the place of the transcendental.

Hence we have the following theorem: Let $f(x)$ be a rational integral function of the fifth or sixth degree in x, and write $f(x_i)=X_i$, $(i=1, 2, 3)$, then the differential equations of the first order with three variables,

$$(\sigma.)\quad \frac{dx_1}{\sqrt{X_1}}+\frac{dx_2}{\sqrt{X_2}}+\frac{dx_3}{\sqrt{X_3}}=0\,;\quad \frac{x_1dx_1}{\sqrt{X_1}}+\frac{x_2dx_2}{\sqrt{X_2}}+\frac{x_3dx_3}{\sqrt{X_3}}=0$$

have two complete algebraic integrals.

This theorem is easily extended to $m-1$ linear differential equations of the first order with m variables, in each of which the variables are separated:

$$(\Sigma.)\quad \frac{x^i{}_1dx_1}{\sqrt{X_1}}+\frac{x^i{}_2dx_2}{\sqrt{X_2}}+\ \ldots\ +\frac{x^i{}_mdx_m}{\sqrt{X_m}}=0,$$
$$(i=0, 1, 2\ \ldots\ m-2),$$

where X_1, X_2, $\ldots$ X_m denote the same integral functions in x_1, x_2, $\ldots$ x_m. Jacobi closes the 'Considerationes generales,' &c. with the remarks: 'We know that Lagrange, starting with the differential equation between two variables [art. 7] came to its complete algebraic integral through direct methods of integration, and so by a new and singular method demonstrated Euler's theorem; and so we think it worth the while to investigate through direct methods of integration the two complete algebraic integrals of the system $(\sigma.)$ above, or more generally the $m-1$ complete algebraic integrals of the system $(\Sigma.)$, and thus adorn Abel's theorem with a new and no less singular demonstration.'

At the end of the 'Note von der geodätischen Linie auf einem Ellipsoid,' &c. ('Werke,' bd. ii. p. 59), Jacobi finds that by making use of a certain substitution he was able to extend the remarkable relation discovered by Legendre between the complete integrals of the first and second kind of two elliptic integrals whose moduli are complements to each other to all hyperelliptic integrals; and this same substitution, he says, led him to the Abelian theorem itself in a way and through considerations which are absolutely different from that of Abel. These considerations originate from a mechanical problem.

The elliptic movement of a planet, or even the motion of a point in a straight line, may be expressed through an equation between two elliptic integrals. We have two methods of treating the same problem, of which the one represents the solution in a transcendental, the other in an

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algebraic form. We thus derive a new method of finding the fundamental theorem for the addition of elliptic integrals. Jacobi says that by generalising this method through the introduction of any number of variables he obtained the general addition-theorem in a new and ready manner; and at the same time there was opened a simpler way, through the application of suitable multiplications, of coming directly to the algebraic integrals of the systems of differential equations ($\sigma.$) and ($\Sigma.$) above. See also a paper by Haedenkamp (Crelle, bd. xxii. p. 184).

(25) Following Jacobi's suggestion of the preceding article, Richelot (Crelle, bd. xxiii. p. 354) extended Lagrange's methods (art. 7) and in a direct manner integrated the differential equations ($\sigma.$) and ($\Sigma.$)[1] above.

By his methods two algebraic integrals are found for Euler's differential equation, of which it is easy to prove that the one is a function of the other; while the two algebraic solutions of ($\sigma.$) are independent. He found two similar solutions for the system ($\Sigma'.$) The general solution of this system of equations he derived in the following form: In system ($\Sigma'.$) let $X=f(x)=A_0+A_1x+A_2x^2+\ldots A_{2n}x^{2n}$, and for brevity write $F(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$ and $F'(x)=\frac{\partial F(x)}{\partial x}$; if, next, the roots of the equation.

$$(a_0+a_1x+a_2x^2+\ldots+a_nx^n)^2-b^2(A_0+A_1x+\ldots+A_{2n}x^{2n})=0$$

be denoted by $x_1, x_2, \ldots x_n, m_1, m_2, \ldots m_n$, and if the coefficients $a_0, a_1, \ldots b$ be determined through the *first* $n+1$ of these roots, then the $n-1$ equations of condition, which must exist in order that the $n-1$ remaining quantities $m_2, m_3 \ldots m_n$ be the roots of this equation, are the $n-1$ complete algebraic integrals of the system ($\Sigma'.$).[2]

Richelot assumed that $f(x)$ has, as factors, $x-a_0$, $x-a_1$, $\ldots$ $x-a_{n-1}$, and for brevity he wrote

$$P(x)=(x-m_2)(x-m_3)\ldots(x-m_n);\ m_1=a_0.$$

[1] Change m into n in the system ($\Sigma.$) in order to have the system adopted by Richelot, which denote by ($\Sigma'.$).

[2] It is interesting to note here certain difficulties that the older mathematicians experienced. Euler (*Inst. Cal. Int.* t. i., cap. vi. § 2, prob. 82, scholion 1) says it is quite clear that transcendents of the form $\int \frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6}}$ cannot be treated in the manner of circular and elliptic integrals; for if the coefficients are restricted so that the root may be extracted, the formula becoming $\int \frac{dx}{\sqrt{a+bx+cx^2+dx^3}}$, it can in no wise happen that several functions of this kind be algebraically compared among one another. Lagrange tried in vain to extend this theorem. This paradoxical thought is easily explained: for two algebraic equations between the arguments of the integrals always satisfy the two transcendental relations between these integrals when the function under the radical sign is of the fifth or sixth degree (art. 24). These algebraic equations exist owing to the fact that the numerator of the integral which is of the first degree has coefficients which are always of such a nature that two of the arguments become roots of a quadratic equation, whose roots involve the other arguments algebraically. As often, therefore, as the transcendents unite within themselves a logarithmic and a trigonometric part, it happens through the algebraic equations that both the trigonometric and the logarithmic parts vanish independently in the two transcendental equations, so that one relation is given among the logarithms and another among the arcs. *Cf.* Richelot (Crelle, bd. ii. p. 181).

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After some reductions he was able to express the $n-1$ integrals of the system $(\Sigma'.)$ in the form,

$$\mathrm{F}(a_h)\left\{\frac{\sqrt{f(x_1)}}{\mathrm{F}'(x_1)}\cdot\frac{1}{a_h-x_1}+\frac{\sqrt{f(x_2)}}{\mathrm{F}'(x_2)}\cdot\frac{1}{a_h-x_2}+\ldots+\frac{\sqrt{f(x_n)}}{\mathrm{F}'(x_n)}\cdot\frac{1}{a_h-x_n}\right\}^2 = -\mathrm{A}_{2n-1}\,\mathrm{P}(a_h),$$
$$(h=1,\,2,\,\ldots\,n-1).$$

One recognises at once much similarity between the quantities employed here and those later used by Weierstrass in Crelle, bd. xlviii.

(26) Jacobi ('Demonstratio nova theorematis Abeliani,' Crelle, bd. xxiv. p. 28), derived the $n-1$ algebraic integrals in a different manner than that given above, and at the same time he established a new proof of Abel's theorem. Instead of the system $(\Sigma.)$ he introduced n differential equations with n variables $\lambda_1, \lambda_2, \ldots \lambda_n$ and the variable t:

$$(\Sigma.'')\quad\begin{cases}\dfrac{\lambda_1^{\,i}\,d\lambda_1}{\sqrt{f(\lambda_1)}}+\dfrac{\lambda_2^{\,i}\,d\lambda_2}{\sqrt{f(\lambda_2)}}+\ldots+\dfrac{\lambda_n^{\,i}\,d\lambda_n}{\sqrt{f(\lambda_n)}}=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (i=0,\,1,\,\ldots\,n-2)\\ \text{with the equation}\\ \dfrac{\lambda_1^{\,n-1}d\lambda_1}{\sqrt{f(\lambda_1)}}+\dfrac{\lambda_2^{\,n-1}d\lambda_2}{\sqrt{f(\lambda_2)}}+\ldots+\dfrac{\lambda_n^{\,n-1}d\lambda_n}{\sqrt{f(\lambda_n)}}=dt\,;\end{cases}$$

where $f(\lambda)$ is an integral function of the $2n-1$th degree in λ.

For brevity let $\mathrm{N}_\kappa=(\lambda_\kappa-\lambda_1)(\lambda_\kappa-\lambda_2)\ldots(\lambda_\kappa-\lambda_n)$, where the vanishing factor $\lambda_\kappa-\lambda_\kappa$ is omitted;

and $y=\sqrt{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_n)}$, where $m-\lambda$ is any factor of the function $f(\lambda)$.

The following lemma is next proved: If $\psi(\lambda)$ is a rational integral function of the $2n-2h$ degree in λ, then

$$\sum_{\kappa=1}^{\kappa=n}\frac{\partial}{\partial\lambda_\kappa}\left(\frac{\psi(\lambda_\kappa)}{\mathrm{N}_\kappa^{\,2}}\right)=0.$$

Making use of this lemma, Jacobi found an algebraic integral of the system $(\Sigma''.)$ in the form

$$y\left(\frac{\sqrt{f(\lambda_1)}}{(m-\lambda_1)\mathrm{N}_1}+\frac{\sqrt{f(\lambda_2)}}{(m-\lambda_2)\mathrm{N}_2}+\ldots+\frac{\sqrt{f(\lambda_n)}}{(m-\lambda_n)\mathrm{N}_n}\right)=\text{Const.}$$

Corresponding to the $2n-1$ factors $m-\lambda$ of the function $f(\lambda)$, there are $2n-1$ integrals of the form just written, of which $n-1$ are sufficient to give the algebraic relations between the n variables $\lambda_1, \lambda_2, \ldots \lambda_n$.

Haedenkamp (Crelle, bd. xxv. p. 178), specialised the general case, and found by geometric considerations the two complete algebraic integrals of the system $(\sigma.)$ above.

(27) In each of the $n-1$ integrals derived by Richelot appear two roots of the equation $f(x)=0$, while in Jacobi's solutions there is found only one; and if imaginary roots enter $f(x)$, the integrals found in both of the methods just given have imaginary forms.

Richelot (Crelle, bd. xxv. p. 97) found another method of solution which also depends upon two roots of the equation $f(x)=0$, but has the property of remaining real when for these two roots any two conjugate roots are substituted. He succeeded further in finding a system of $n-1$

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solutions of the system (Σ'.), which contain none of the roots of the equation $f(x) = 0$, or presupposes them in any manner. This more general solution is an extension of Jacobi's method, which was effected by the consideration of mechanical problems (art. 24) and throws new light upon Abel's theorem, of which some fundamental forms are derived through suitable integration.

Writing $\mathrm{F}(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$, Richelot found as an integral of Σ'.

$$\text{Const.} = \mathrm{F}(a)\left\{\frac{\sqrt{f(x_1)}}{(x-x_1)\,\mathrm{F}'(x_1)} + \ldots + \frac{\sqrt{f(x_n)}}{(x-x_n)\,\mathrm{F}'(x_n)}\right\}^2 - \frac{f(a)}{\mathrm{F}(a)} - \mathrm{A}_{2n}\mathrm{F}(a),$$

in which $f(x) = \mathrm{A}_0 + \mathrm{A}_1 x + \ldots + \mathrm{A}_{2n}x^{2n}$, and a denotes any arbitrary integer.

Let a take $n - 1$ different values a_1, a_2, a_{n-1}, and we have a complete system of $n-1$ solutions of the equations (Σ'.) Through suitable integrations of (Σ'.) different forms of Abel's theorem are derived in accordance with Jacobi's suggestions in art. 24.

(28) Jacobi (Crelle, bd. xxxii. p. 220) found that the $n - 1$ algebraic equations through which the system (Σ'.) was integrable consist of one equation of the second degree in a_1 and a_2 (where a_1 denotes the *sum* of the quantities $x_1, x_2, \ldots x_n$, a_2 the sum of these quantities taken two at a time, a_3 the sum taken three at a time, etc.), and of $n - 2$ equations by means of which $a_3, \ldots a_n$ are linearly expressed in terms of a_1 and a_2; and further, that between any two of the quantities a there exists a quadratic, and between any three a linear relation (*cf.* also Weierstrass, 'Math. Werke,' bd. i. p. 267).

Jacobi further showed that if we write

$$\begin{aligned} f(x) = {} & (bx^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n)^2 \\ & + (cx^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n)^2 \\ & - (ax^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n)^2, \end{aligned}$$

the differential equations (Σ'.) are completely integrated, if for $x_1, x_2, \ldots x_n$ are written the roots of the equation

$$\begin{aligned} & x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n \\ & - (bx^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n)\cos\phi \\ & + (cx^n + c_1x^{n-1} + c_2x^{n-2} + \ldots + c_n)\sin\phi, \end{aligned}$$

where ϕ denotes a variable angle.

See also Brioschi (Crelle, bd. lv. p. 56); and Cayley ('Camb. and Dubl. Math. Journ.' vol. iii. 1848, p. 116); 'Math. Papers,' vol. i. p. 366.

(29) *Reduction and Transformation of hyperelliptic integrals.*—We noted Landen's substitutions for the elliptic integrals in art. 8.

Legendre, in the thirty-second chapter of the 'Traité des fonctions elliptiques,' t. i. p. 254, showed in general how to reduce the integrals

$$\int \frac{\mathrm{P}dx}{\sqrt{\beta + \gamma x^2 + \delta x^4 + \gamma x^6 + \beta x^8}},$$

where P denotes a rational function in x, to elliptic integrals.

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In this connection we note a paper by Richelot, 'Ueber die Reduction des Integrales $\int \frac{f(x)dx}{\sqrt{\pm(1-x^8)}}$ auf elliptische Integrale,' Crelle, bd. xxxii. p. 213.

In the third supplement of the 'Traité des fonctions elliptiques,' p. 207, Legendre investigated certain special forms of Abelian integrals as $\int \frac{dx}{\sqrt{1 \pm x^5}}$, and calculated the values of such integrals with fixed limits : on p. 333 (*loc. cit.*), by making use of certain substitutions, he showed that the integral $\int_0^x \frac{dx}{\sqrt{x(1-x^2)(1-\kappa^2x^2)}}$ could always be reduced to two elliptic integrals of the first kind that have the same amplitude, and whose moduli are the complements of each other (*cf.* a memoir by Catalan, that was crowned by the Academy of Brussels, 'Mém. couronnés par l'Acad. Royale de Brux.' xiv. 2nde partie, p. 2).

Jacobi (Crelle, bd. viii. p. 416) extended this theorem to the integrals $\int \frac{xdx}{\sqrt{(x+x^2)(1-\kappa^2x^2)}}$, and to integrals of the more general form

$$\int \frac{dx}{\sqrt{R(x)}} \text{ and } \int \frac{xdx}{\sqrt{R(x)}},$$

where $R(x) = x(1-x)(1-\kappa\lambda x)(1+\kappa x)(1+\lambda x)$.

He showed that such integrals may be always expressed as the sum of two elliptic integrals of the first kind which have the same amplitude, but in general different moduli. See also Gauss, 'Determinatio attractionis, quam in punctum quodvis positionis datae exercet planeta,' &c. ('Werke,' iii. p. 333).

More recent examples[1] of similar reductions are given by Hermite ('Ann. Soc. Scient. Brux.' I., B, p. 1, 'Comptes Rendus,' t. xl. 1855) : John C. Mallet ('Trans. of Dublin,' 1874) (he extends the theorems of Jacobi to hyperelliptic integrals of any kind) ; J. C. Mallet (Crelle, bd. lxxvi. p. 79, and bd. lxxix. p. 176) ; Cayley ('Compt. Rend.' t. lxxxv. pp. 265, 373, 426, 472).

(30) Richelot (Crelle, bd. xii. p. 181) shows that integrals of the form

$$\text{(I.)} \qquad \int \frac{F(x)dx}{\sqrt{(A+Bx+Cx^2)(A_1+B_1x+C_1x)(A_2+B_2x+C_2x^2)}},$$

[1] We mention in passing Gordan, 'Ueber die Invarianten binärer Formen höheren Transformationen' (Crelle, bd. lxxi. p. 164) ; Arondhold, 'Integration irrationaler Differentiale' (Crelle, bd. lxi. p. 95). In this paper extensive use is made of invariants. Brioschi (*Compt. Rend.* t. lvi. and t. lix.) bases this theory of the reduction of integrals $\int F(x,y)dx$ upon the theory of the covariants of the ternary form. See also Brioschi (*Compt. Rend.* lxxxv. p. 708, 1877) ; Königsberger (Crelle, bd. lxiv. p. 17 ; bd. lxv. p. 335 ; bd. lxvii. p. 97 ; bd. lxvii. p. 56 ; bd. lxxxv. p. 273 ; bd. lxxxix. p. 89 ; *Math. Ann.* bd. xv. p. 174) ; Bolza (*Math. Ann.* bd. xxviii. p. 447). A somewhat extended account of the reduction of hyperelliptic integrals, including many of the more recent investigations, is found in Enneper's *Elliptische Functionen*, p. 501 *et seq.*

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where $F(x)$ denotes a rational function in x, and where the factors in the denominator are resolvable into real linear factors, may be reduced by means of twelve substitutions of the form $x=\frac{a+bz^2}{c+dz^2}$ into an aggregate of integrals

$$\int_0^z \frac{\psi(z^2)dz}{\sqrt{(1-z^2)(1-\kappa^2z^2)(1-\lambda^2z^2)(1-\mu^2z^2)}},$$

where $\phi(z^2)$ is a rational function in z^2, z^2 being in value situated between 0 and 1, as are also the quantities κ^2, λ^2 and μ^2.

Richelot further proved that, whatever the degree of the function under the square-root sign, provided it consist of linear real factors, integrals corresponding to (I.) may be reduced to a form similar to that just written.[1]

He further divides these integrals into the three principal kinds, and by means of Abel's theorem considers the peculiar properties of the respective kinds. By making use of irrational substitutions which depend upon a quadratic equation, he finds that integrals of the form

$$\int \frac{(M+Nz^2)dz}{\sqrt{(1-z^2)(1-\kappa^2z^2)(1-\lambda^2z^2)(1-\mu^2z^2)}}$$

may be reduced to the same form

$$\int \frac{(M_1+N_1y^2)dy}{\sqrt{(1-y^2)(1-\kappa'^2y^2)(1-\lambda'^2z^2)(1-\mu'^2z^2)}},$$

where the moduli are either greater or less than the old moduli. By repetition of this procedure the moduli rapidly approach zero or unity (*cf.* art. 8).

(31) In a later paper Richelot (Crelle, bd. xvi. p. 221) reduces integrals of the form $\int_0^\infty \frac{dx}{\sqrt{1+x^6}}$ to $\frac{1}{2}\int_0^1 \frac{2dz}{\sqrt{(1-z^2)(1-\cos^2\frac{\pi}{10}z^2)(1-\cos^2\frac{3\pi}{10}z^2)}}$ by means of the transformations $x=\frac{1+t}{1-t}$ where $t=\pm\sqrt{1-z^2}$.

Making use of a substitution that was suggested by Jacobi,

$$2y=\sqrt{a+bz+cz^2}\pm\sqrt{a-bz+cz^2},$$

he shows that the sum and the difference of the integrals

$$\int \frac{dx}{\sqrt{(a+bx+cx^2)(x^2+d)(x^2+e)}} \text{ and } \int \frac{dx}{\sqrt{(a-bx+cx^2)(x^2+d)(x^2+e)}}$$

may be expressed through *one* Abelian integral. The second part of this memoir is devoted to the numerical calculation of hyperelliptic integrals of the first order.

In the posthumous writings of Jacobi a method is given whereby the

[1] *Cf.* Jacobi, *Gesam. Werke*, bd. ii. p. 38. These integrals may be expanded in converging series according to sines or cosines of multiples of the same angle.

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hyperelliptic integrals of the first order may be reduced into canonical forms by means of certain substitutions, even when the factors of the function under the square-root sign in the denominator of the integrand are not all real. Jacobi also avoids the imaginary arguments introduced by Richelot through the application of the substitution $2y=\sqrt{a+bz+cz^2}\pm\sqrt{a-bz+cz^2}$, and which Richelot again reduced to real arguments by means of Abel's theorem. When the degree of the function under the root sign is greater than the sixth, and when this function contains imaginary factors, Jacobi asserts that no one has found the substitutions by which the reduction may be performed. At the close of this article Jacobi discusses Euler's addition-theorem from a more general standpoint than that taken by Lagrange (art. 7).

(32) *Addition-theorems for hyperelliptic integrals.*—Jacobi (Crelle, bd. xxx. p. 121) derived an interesting form of the addition-theorem for hyperelliptic integrals of the second and third kinds. Let R be a given integral function of the $2n$th degree in x

$$R=a_1x^{2n}+a_2x^{2n-1}+\ldots+1,$$

and V an integral function of the nth degree with unity as coefficient of highest power of x; further, let a be a constant and

$$(1)\quad xV^2+a^2R=(x-x_1)(x-x_2)\ldots(x-x_{2n+1})$$
$$=x^{2n+1}+a_1x^{2n}+a_2x^{2n-1}+\ldots+a_{2n+1};$$

then it may be proved that [1]

$$(I.)\quad \sum_{i=1}^{i=2n+1}\int\frac{x_i^m dx_i}{\sqrt{x_iR(x_i)}}=0,$$

where m takes any of the values $0, 1, \ldots n-1$, and where the upper and lower limits of the integrals are two systems of roots of two equations of the form (1), in which a and the coefficients of V have different values, while R remains the same.

Further, we may deduce the following expressions

$$(II.)\quad \sum_{i=1}^{i=2n+1}\int\frac{x_i^{n+\kappa}dx_i}{\sqrt{x_iR(x_i)}}=\Lambda_\kappa,$$

where Λ_κ ($\kappa=0, 1, \ldots n+1$) are algebraic functions of $a, a_1, a_2, \ldots a_{2n+1}$. Jacobi gives a rule by which those functions are easily determined.

Jacobi makes the theorem above more general by the introduction of a new variable x_{2n+2}, and the formation of new formulæ

$$(III.)\quad \sum_{i=1}^{i=2n+2}\int\frac{x_i^{n+\kappa}dx_i}{\sqrt{x_iR(x_i)}}=\Lambda_\kappa,$$

where the quantities $a, a_1, \ldots a_{2n+1}, a_{2n+2}$ are now determined through an equation corresponding to (1)

$$(x-x_1)(x-x_2)\ldots(x-x_{2n+2})=x^{2n+2}+a_1x^{2n+1}+a_2x^{2n}+\ldots+a_{2n+2}.$$

For the integrals of the third kind Jacobi proves the following theorem :—

By means of $n+2$ given quantities $x_1, x_2, \ldots x_{n+1}, a$, let three

[1] See Abel, *Œuvres*, t. i. p. 444.

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systems of n quantities $w_1, w_2, \ldots w_n$; $y_1, y_2, \ldots y_n$; $z_1, z_2 \ldots z_n$ be determined through the three systems of n transcendental equations

$$\sum_{i=1}^{i=n}\int\frac{w_i^m dw_i}{\sqrt{w_i \mathrm{R}(w_i)}} = \sum_{i=1}^{i=n+1}\int\frac{x_i^m dx_i}{\sqrt{x_i \mathrm{R}(x_i)}},$$

$$\sum_{i=1}^{i=n}\int\frac{y_i^m dy_i}{\sqrt{y_i \mathrm{R}(y_i)}} = \sum_{i=1}^{i=n+1}\int\frac{x_i^m dx_i}{\sqrt{x_i \mathrm{R}(x_i)}} + \int\frac{a^m da}{\sqrt{a \mathrm{R}(a)}},$$

$$\sum_{i=1}^{i=n}\int\frac{z_i^m dz_i}{\sqrt{z_i \mathrm{R}(z_i)}} = \sum_{i=1}^{i=n+1}\int\frac{x_i^m dx_i}{\sqrt{x \mathrm{R}(x_i)}} - \int\frac{a^m da}{\sqrt{a \mathrm{R}(a)}},$$

$$(m=0, 1, \ldots n-1)$$

in which $\mathrm{R}(x)$ is a given function of x of the $2n$th degree; then there exists among the integrals of the third kind the following equation :—

$$\text{(IV.)}\quad \sum_{i=1}^{i=n}\int\frac{dw_i}{(a-w_i)\sqrt{w_i \mathrm{R}(w_i)}} = \sum_{i=1}^{i=n+1}\int\frac{dx_i}{(a-x_i)\sqrt{x_i \mathrm{R}(x_i)}} - \frac{1}{\sqrt{a \mathrm{R}(a)}} \log \frac{1+(-1)^n\sqrt{\dfrac{x_1 x_2 \ldots x_{n+1}}{a y_1 y_2 \ldots y_n}}}{1+(-1)^n\sqrt{\dfrac{x_1 x_2 \ldots x_{n+1}}{a z_1 z_2 \ldots z_n}}}.$$

When n is made unity in the above formulæ, they reduce to Legendre's form of the addition-theorems for elliptic functions.[1] See Jacobi, 'Extrait d'une lettre addressée à M. Hermite' (Jacobi's 'Werke,' ii. p. 120).

(33) *Interchange of parameter and argument of the integrals of the third kind.*—Legendre discovered this remarkable property of elliptic integrals, and derived other formulæ in the same connection. ('Ex. de Calc. Int.' t. i. p. 134 *et seq.*) The results of Legendre are implicitly contained in the following formula, due to Abel ('Œuvres,' t. ii. p. 43) :—

$$\sqrt{\theta(a)}\int\frac{dx}{(x-a)\sqrt{\varphi(x)}} - \sqrt{\theta(x)}\int\frac{da}{(a-x)\sqrt{\varphi(a)}}$$

$$= \sum \tfrac{1}{2}(n-m)\, a_{m+n+2}\int\frac{x^n dx}{\sqrt{\varphi(x)}}\int\frac{a^m da}{\sqrt{\varphi(a)}},$$

where $\phi(x)$ is any integral function of x, a_{m+n+2} are constants, and m,n integers. Jacobi (Crelle, bd. xxxii, p. 185; 'Werke,' bd. ii. p. 123) obtained the analogous formula. Let $f(x)$ be a rational integral function of x, and $f_1(x)$ and $f_2(x)$ any two rational integrals of x, whose sum

$$f_1(x)+f_2(x) = \frac{df(x)}{dx};$$

[1] In this connection see Heine (Crelle, bd. lxi. p. 276); Schumann (*Math. Ann.* bd. vii. p. 623); Scheibner (*Math. Ann.* bd. xxxiv. p. 473.) We mention here a paper by Serret, 'Mémoire sur la représentation géométrique des fonctions elliptiques et ultra-elliptiques' (*Liour. Journ.* t. x. pp. 257, 286, 351 and 421). Liouville (*Compt. Rend.* xxi. p. 1,255, or *Liour. Journ.* t. x. p. 456) gives a method of representing elliptic and hyperelliptic curves. See Ellis' report, p. 72.

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further write

$$\frac{d\log\phi(x)}{dx}=\frac{f_1(x)}{f(x)},\quad \frac{d\log\psi(x)}{dx}=\frac{f_2(x)}{f(x)},$$

then

$$\phi(a)\int\frac{dx}{(x-a)\,\phi(x)}-\psi(x)\int\frac{da}{(a-x)\,\psi(a)}$$

is equal to an aggregate of products of the form

$$C_{m,n}\int\frac{a^m da}{\psi(a)}\int\frac{x^n dx}{\phi(x)},$$

where m and n are integers, and the quantities $C_{m,n}$ are constants.

By making use of certain theorems from the theory of linear differential equations, Abel and Jacobi extended the above results. Formulæ are derived in which the products of the sums of n such integrals remain unchanged when the argument and parameter are interchanged; and by means of these formulæ n^2 integrals may be linearly expressed through n^2 other integrals, in which the parameter and argument have been interchanged; and *vice versâ*, these n^2 integrals may in turn be expressed linearly through the first n^2 integrals. See in this connection Clebsch and Gordan, 'Theorie der Abelschen Functionen,' p. 114, where some interesting consequences are deduced.[1]

(34) Abel, 'Sur l'intégration de la formule différentielle $\frac{\rho\,dx}{\sqrt{R}}$, R et ρ étant les fonctions entières' ('Œuvres,' t. i. p. 104), gave the conditions under which the integrals $\int\frac{\rho\,dx}{\sqrt{R}}$ may be expressed through functions of the form $\log\frac{p+q\sqrt{R}}{p-q\sqrt{R}}$, where p and q are integral functions.[2] In another memoir, published after his death ('Œuvres,' t. ii. p. 87), the general problem is solved, when may an elliptic integral be reduced to algebraic-logarithmic functions? Weierstrass ('Math. Werke,' bd. i. p. 227) says that the general problem of integrating an algebraic differential by means of logarithms, in so far as this is possible, was first proposed by Abel, who had arrived at very important results, as is seen from a letter written to Legendre ('Œuvres,' t. ii. p. 271), and it is very probable that just these investigations led him to his celebrated theorem.

Abel ('Œuvres,' t. i. p. 549) derived the more general theorem relative to the form which one must give to the integral of any algebraic function when it is possible to express this integral by means of algebraic and logarithmic functions and elliptic integrals:

Let $y_1, y_2, \ldots y_\mu$ be algebraic functions of $x_1, x_2, \ldots x_\mu$, and let the x's be connected by any number of algebraic equations.

[1] *Cf.* also Weierstrass, *Werke*, bd. i. p. 113, where the source of the property of interchange of parameter and argument is revealed; Frobenius (Crelle, bd. lxxiii. p. 93).

[2] *Cf.* papers by Tschebyscheff (*Liouv. Journ.* 2nde série, t. ii. p. 1); Pick (*Sitzungsb. der kaiserl. Akad. der Wissenschaften in Wien*, 1882, p. 643); Plana (Crelle, bd. xxxvi. p. 1).

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If any integral of the form

$$\int (y_1 dx_1 + y_2 dx_2 + \ldots + y_\mu dx_\mu)$$

is expressible in algebraic and logarithmic functions and elliptic integrals in such a way that

$$\int (y_1 dx_1 + y_2 dx_2 + \ldots + y_\mu dx_\mu) = u + A_1 \log v_1 + A_2 \log v_2 + \ldots + A_\nu \log v_\nu + a_1 \psi_1(t_1) + a_2 \psi_2(t_2) + \ldots + a_n \psi_n(t_n),$$

where $A_1, A_2 \ldots, a_1, a_2, \ldots$ are constants, $u, v_1, v_2, \ldots t_1, t_2, \ldots$ are algebraic functions of $x_1, x_2, \ldots$ and $\psi_1, \psi_2, \ldots$ are any elliptic integrals of the three kinds with any moduli and parameters, then Abel proved that this integral may be always expressed in the form :

$$\delta \int (y_1 dx_1 + y_2 dx_2 + \ldots y^\mu dx_\mu) = r + A' \log \rho' + A'' \log \rho'' + \ldots + A^{(\kappa)} \log \rho^{(\kappa)} + a_1 \psi_1(\theta_1) + a_2 \psi_2(\theta_2) + \ldots + a_n \psi_n(\theta_n),$$

where δ is an integer, $a_1, a_2, \ldots a_n$ are the same as in the preceding expression ; $A', A'' \ldots A^{(\kappa)}$ are constants ; $\theta_1, \Delta_1(\theta_1), \theta_2, \Delta_2(\theta_2) \ldots \theta_n, \Delta_n(\theta_n)$; $r, \rho', \rho'', \ldots \rho^{(\kappa)}$ are rational functions [1] of the quantities $x_1, x_2, \ldots x_\mu$; $y_1, y_2, \ldots y_\mu$.

Abel remarks (p. 550) that this theorem is not only fundamental in all that concerns the application of algebraic and logarithmic functions and elliptic integrals to the theory of the integration of *algebraic* differentials, but it includes all the possible reductions of the integrals of algebraic formulæ by the aid of algebraic and logarithmic functions.

As a corollary is the following theorem : If $\int \frac{\rho dx}{\Delta(x, c)}$, where ρ is any rational function of x, and $\Delta(x,c)$ denotes $\pm\sqrt{(1-x^2)(1-c^2x^2)}$, is expressible by algebraic and logarithmic functions and elliptic integrals, then we may always suppose that

$$\int \frac{\rho dx}{\Delta(x, c)} = p\Delta(x, c) + a\psi(y) + a'\psi_1(y_1) + a''\psi_2(y_2) + \ldots + A_1 \log \frac{q_1 + q_1'\Delta(x, c)}{q_1 - q_1'\Delta(x, c)} + A_2 \log \frac{q_2 + q_2'\Delta(x, c)}{q_2 - q_2'\Delta(x, c)} + \ldots,$$

where all the quantities $p, q_1, q_2, \ldots q_1', q_2', \ldots y, y_1, y_2, \ldots$ are rational functions of x.

From this may be derived a complete solution of the equation

$$\frac{dy}{\Delta(y, c)} = \epsilon \frac{dx}{\Delta(x, c')}$$

where ϵ is a constant, and whence also the general transformation of elliptic integrals of the first kind.

(35) Similar problems were discussed by Liouville [2] ('Mémoires des Savants Étrangers,' t. v. pp. 76 and 103) before he had seen the methods used by Abel. Liouville says that the problems proposed by him do not differ in their origin from those enunciated by Lagrange in the 'Théorie

[1] $\psi_m(x) = \int \frac{\theta' dx}{\Delta_m(x)}$, when θ' denotes any rational function of x, and $\Delta_m(x) = \pm\sqrt{(1-x^2)(1-c_m^2x^2)}$.

[2] See Poisson's report on these memoirs in Crelle, bd. xii. p. 342, and the note appended by Liouville ; also Jürgensen (Crelle, bd. xxiii. p. 129).

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analytique des Probabilités,' viz. that the integrals of differential functions cannot contain other radical quantities than those that enter these functions, theorems which were known to the first inventors of analysis (see Leibnitz, 'Act. erud. Lips.'). In the first of the memoirs mentioned above Liouville proposes the problem of finding the form of the integral $\int y dx$, when this integral may be expressed algebraically, where y and x are connected by an algebraic equation

$$y^n - Ly^{n-1} - \ldots - My - N = 0,$$

L, M, . . . designating rational functions of x; he shows that the value of such an integral is equal to a certain rational function of x and y. In the discussion of this theorem he classifies functions of one or more variables according to the irrationalities that enter them.[1]

In the second memoir the general theorem which he proposes to demonstrate is: if any algebraic explicit or implicit function y is given, it is always possible to decide if it has or has not for an integral an explicit or implicit algebraic function; and if the question is decided in the affirmative, the same process will give the value $\int y dx$.

He shows that if the integral $\int y dx$ may be expressed algebraically, it has a value of the form:

$$\int y dx = \alpha + \beta y + \gamma y^2 + \ldots + \lambda y^{n-1},$$

in which α, β, γ, . . . λ are rational functions of x.

In the twenty-third volume of the 'Journ. de l'École Poly.' p. 37, Liouville finds that if the integral $\int y dx$ is expressible as an explicit finite function of x, it must be of the form:

$$\int y dx = t + A \log u + B \log v + \ldots + C \log w,$$

where A, B, . . . are constants, and t, u, v, . . . are algebraic functions of x. This theorem is of course contained in the one of Abel in the preceding article. Liouville further shows that if $\int \frac{P dx}{\sqrt{R}}$, P and R denoting integral polynomials, cannot be expressed by an algebraic function of x, it cannot be expressed as a finite explicit function of x; from this follows that an elliptic integral of either the first or second kind cannot be expressed as an explicit function of its variable. (See also Liouville, 'Liouv. Journ.' t. v. pp. 34 and 441, where it is proved that the same integrals, considered as functions of their modulus, cannot be expressed in finite form.[2])

(36) Jacobi ('De functionibus duarum variabilium quadrupliciter periodicis,' Crelle, bd. xiii. p. 55; 'Werke,' bd. ii. p. 25) proved that any one-valued function of one variable cannot have more than two independent periods, and that the ratio of these two periods cannot be a real quantity and is irrational.

[1] See also two memoirs by Liouville, 'Sur la Classification des Transcendantes' (*Liouv. Journ.* t. ii. p. 56, and t. iii. p. 523); and Poisson (Crelle, bd. xii. p. 89, and bd. xiii. p. 93)

[2] Cf. *Liouv. Journ. de l'École Poly.*, t. xiv. p. 137; and Ellis, p. 70.

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Making use of Richelot's transformation, Jacobi wrote the general integral under consideration in the form :

$$(A.) \qquad \int_0^x \frac{(\alpha+\beta x)dx}{\sqrt{x(1-x)(1-\kappa^2 x)(1-\lambda^2 x)(1-\mu^2 x)}}.$$

He distinguished the values of this integral within the six intervals :

$$(1)\ -\infty \ldots 0, \quad (2)\ 0 \ldots 1, \quad (3)\ 1 \ldots \frac{1}{\kappa^2},$$

$$(4)\ \frac{1}{\kappa^2} \ldots \frac{1}{\lambda^2}, \quad (5)\ \frac{1}{\lambda^2} \ldots \frac{1}{\mu^2}, \quad (6)\ \frac{1}{\mu^2} \ldots \infty.$$

When the upper limit x in formula (A.) is considered as a function of u and written $x=\lambda(u)$, this function behaves like a periodic function within each of these intervals, and therefore seems to have six periods, of which four are independent. Further, this function remains unchanged when u assumes any real or imaginary value, or better expressed, of the values which u may take, there are always those which differ from any real or imaginary quantity by less than any assignable quantity, however small.

Jacobi found in this a troublesome paradox, which, however, he had already in a measure overcome by means of Abel's theorem (see following article).

Jacobi next proved that if a given function of two variables is an one-valued function of these variables, it is impossible for this function to have more than four independent periods.[1]

(37) *The inverse functions.*—Corresponding to the function $x=\mathrm{sn}\ u$ (art. 9), if we try to introduce into analysis a transcendent $x=\lambda(u)$, where

$$(I.)\ u=\int_0^x \frac{dx}{\sqrt{X}}=\Pi(x),$$

X being a rational integral function of the fifth or sixth degree in x, then there is no analogy between this function and the elliptic function $x=\mathrm{sn}\ u$, since such a function, as seen above, has for every value of u not only many values, but is wholly indeterminate,[2] if for the definition of the integral we consider only the limits and not the paths which the variable describes from one limit to the other. Hence, when we consider the integral (I.) by itself, its inversion does not give useful results.

The close connection between the integral $\Pi(x)$ and the integral

$$(II.)\ \Pi_1(x)=\int_0^x \frac{x dx}{\sqrt{X}}$$

was seen in art. 23.

Jacobi conceived the very fortunate idea of inverting these integrals

[1] The more general theorem that *a one-valued function of* n *variables cannot have more than* 2n *independent periods* was proved much later by Riemann (Crelle, bd. lxxi. p. 197). See also Weierstrass (*Monatsb. der Akad. der Wiss. zu Berlin*, 1876, p. 680; *Functionenlehre*, p. 166).

[2] *Cf.* Jacobi (Crelle, bd. ix. p. 394; *Werke*, bd. ii. pp. 7 and 516).

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by connecting two integrals u and v with the variables x_1 and x_2 in the following equations :

$$(1)\ u=\int_0^{x_1}\frac{dx}{\sqrt{X}}+\int_0^{x_2}\frac{dx}{\sqrt{X}}=\Pi(x_1)+\Pi(x_2),$$

$$(2)\ v=\int_0^{x_1}\frac{xdx}{\sqrt{X}}+\int_0^{x_2}\frac{xdx}{\sqrt{X}}=\Pi_1(x_1)+\Pi_1(x_2).$$

In these two equations, when u is given, the upper limits x_1 and x_2 are not yet determined, there being two of them ; so that we may regard u and v as independent of each other. This we cannot do when we consider the integrals (I.) and (II.) separately ; for if x was determinate for a given value of u in (I.), then it would be determined in (II.), and therefore (II.) would be determined.

When u and v are given, the totality of the upper limits (that is, of x_1 and x_2) is known ; but these quantities may be permuted, so that x_1+x_2 and $x_1.x_2$ are definitely determined when u and v are given, and may be expressed as the roots of a quadratic equation $Ax^2+Bx+C=0$, where $x_1+x_2=-\frac{B}{A}=\phi(u,v)$, and $x_1.x_2=\frac{C}{A}=\psi(u,v)$. A, B, and C are functions of u and v, which have definite finite values for all finite values, real or imaginary, of the two arguments u and v ; and the functions $\phi(u,v)$ and $\psi(u,v)$ have with reference to the arguments u and v four simultaneous independent periods.

Let the values of x_1 and x_2, determined from the quadratic equation above, be $x_1=\lambda(u,v)$, $x_2=\lambda_1(u,v)$.

In these functions it is seen that when one of the arguments goes to infinity, the other becomes indeterminate, and when one of the arguments changes by a constant quantity the other argument is also changed, so that both arguments undergo an alteration at the same time, and the period of one argument is determined by the period of the other ; this is the characteristic property of the periodicity.

(38) The functions $\lambda(u,v)$ and $\lambda_1(u,v)$ are analogous to the elliptic and trigonometric functions, and may be algebraically expressed in terms of functions that contain only one variable.[1]

For let x_1^0 and x_2^0 be the values of x_1 and x_2 when we put $v=0$, and $x_1^{(0)}$ and $x_2^{(0)}$ the values of these variables when $u=0$.

Then from equation (1) and (2) above

$$\Pi(x_1^0)+\Pi(x_2^0)=u\ ;\ \Pi_1(x_1^0)+\Pi_1(x_2^0)=0,$$
$$\Pi(x_1^{(0)})+\Pi(x_2^{(0)})=0\ ;\ \Pi_1(x_1^{(0)})+\Pi_1(x_2^{(0)})=v.$$

Hence

$$\Pi(x_1^0)+\Pi(x_2^0)+\Pi(x_1^{(0)})+\Pi(x_2^{(0)})=u,$$
$$\Pi_1(x_1^0)+\Pi_1(x_2^0)+\Pi_1(x_1^{(0)})+\Pi_1(x_2^{(0)})=v.$$

Owing to Abel's theorem, the two quantities x_1 and x_2 may be algebraically expressed as functions of x_1^0, x_2^0, $x_1^{(0)}$, and $x_2^{(0)}$ in such a way that

$$\Pi(x_1^0)+\Pi(x_2^0)+\Pi(x_1^{(0)})+\Pi(x_2^{(0)})=\Pi(x_1)+\Pi(x_2)=u,$$
$$\Pi_1(x_1^0)+\Pi_1(x_2^0)+\Pi_1(x_1^{(0)})+\Pi_1(x_2^{(0)})=\Pi_1(x_1)+\Pi_1(x_2)=v$$

[1] See Jacobi (Crelle, bd. xxx. p. 183 ; *Werke*, bd. ii. p. 85).

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exist simultaneously; so that x_1 and x_2 are algebraic functions of the four quantities x_1^0, x_2^0, $x_1^{(0)}$, and $x_2^{(0)}$; that is, the functions of two variables $\lambda(u,v)$ and $\lambda_1(u,v)$ may be algebraically expressed in terms of the four quantities of a single variable

$$\lambda(u,0),\ \lambda_1(u,0),$$
$$\lambda(0,v),\ \lambda_1(0,v).$$

Eisenstein (Crelle, bd. xxvii. p. 185) writes as follows regarding the inverse functions: 'Great difficulties are found with the Abelian integrals whose inverse functions have a triple or multiple periodicity. Under the assumption that an integral with definite lower limit may take all possible real and imaginary values for any given value of the variable, the Abelian integral ceases to be a function of its variable. In order to meet these difficulties, for example in the Abelian integrals of the first order, Jacobi considered two such integrals connected by the relations (1) and (2) of the preceding article. But if we grant that the function $\Pi(x_1)$ can have all possible values for any given value of x_1, the function $\Pi(x_2)$ may have the same property for every given value of x_2, and so the sum u may for a greater reason take all possible values for given values of x_1 and x_2. The same is true of v; so that it is not clear how we may speak in this wise of a dependency between u, v, x_1, and x_2.' Eisenstein then proposes, in order to set forth the real nature of these functions after the analogue of the elliptic functions (art. 11), to form the quotients of the quotients of infinite triple products.

Jacobi ('Werke,' bd. ii. p. 86) corrects Eisenstein's objections with the remarks that Eisenstein did not understand the nature of the functions $\lambda(u,v)$, $\lambda_1(u,v)$, his mistake being that he did not sufficiently comprehend the fundamental principle of the co-existence of the periods relative to the two arguments u and v. He then asks if the quotients of quotients are not simply quotients, and points out how Eisenstein has made some fundamental mistakes in the theory of elliptic functions (Crelle, bd. xxvii. pp. 185 and 285).

(39) Hermite (Extrait d'une Lettre à M. Liouville, 'Comp. Rendus,' t. xviii.; 'Liouville's Journ.' t. ix. p. 353) introduces into the analysis of the transcendents of any algebraic differentials the inverse functions of several variables, after the example of that which had been done by Jacobi for the hyperelliptic integrals of the first order.

Using the notation of Abel and of Minding, he takes

$$\chi(y)=p_0+p_1y+p_2y^2+\ \ldots\ +p_{n-1}y^{n-1}+y^n=0,$$

an irreducible algebraic equation, whose coefficients are rational and integral functions of x. The roots of this equation he denotes by y_1, y_2, $\ldots\ y_n$. He further writes:

$$f(x,y)=t_0+t_1y+\ \ldots\ +t_{n-2}y^{n-2}$$

any rational integral function in x and y, in which the degrees of the x's are subjected to certain restrictions.

Finally, let γ denote the number of arbitrary constants that are contained in the function $f(x,y)$; this function may then take γ different forms, which are represented by

$$f_1(x,y), f_2(x,y), \ \ldots\ f_\gamma(x,y).$$

Designate by $x_1, x_2, \ldots\ x_\mu\ \ldots\ \mu$ variables where $\mu>\gamma$ and by $y_{(1)}, y_{(2)}$ $\ldots\ y_{(\omega)}$ irrational functions arbitrarily chosen among the n roots $y_1, y_2,$ $\ldots\ y_n$.

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Then by Abel's theorem we have in algebraic form the complete integrals of the system of equations.

$$\sum_{j=1}^{j=\mu} \int \frac{f_i(x_j, y_{(j)})}{\chi'(y_{(j)})}\, dx_j = 0,$$
$$(i=1, 2, \ldots \gamma).$$

(40) Hermite then takes for the inverse functions the quantities $x_1, x_2, \ldots x_\gamma$ defined by the γ equations

$$\sum_{j=1}^{j=\mu} \int_0^{x_j} \frac{f_i(x_j, y_{(j)})}{\chi'(y_{(j)})}\, dx_j = u_i,$$
$$(i=1, 2, \ldots \gamma),$$

and writes

$$x_i = \lambda\,(u_1, u_2, \ldots u_\gamma),$$
$$(i=1, 2, \ldots \gamma).$$

It follows [1] without difficulty that the γ functions

$$\lambda\,(u_1+v_1, u_2+v_2, \ldots u_\gamma+v_\gamma),$$
$$(i=1, 2, \ldots \gamma),$$

are the roots of an algebraic equation of the degree γ, whose coefficients are rational functions of the different functions

$$\lambda_i\,(u_1, u_2, \ldots u_\gamma),\ \lambda_i\,(v_1, v_2, \ldots v_\gamma),$$
$$(i=1, 2, \ldots \gamma).$$

In the third section of this memoir, Hermite discusses the periodic properties of these functions, and determines their periods.

The theorem relative to the addition of the arguments leads to the expression of the inverse functions in all their generality in terms of the simplest particular functions, in which we may suppose successively that only one argument varies, the others being constant, zero for example (*cf.* art. 38).

As an illustration, we saw in art. 24 that functions connected with the hyperelliptic integrals of the second order arise, in which appear three arguments, $\Pi(u, v, w)$, say; and from Abel's theorem it follows that

$$\Pi(u+u'+u'', v+v'+v'', w+w'+w'') = \Pi(u, v, w) + \Pi(u', v', w') + \Pi(u'', v'', w'') + \text{alg. and logarith. function.}$$

Now writing

$$u=0, v=0;\ u'=0, w'=0;\ v''=0, w''=0,$$

we have

$$\Pi(u'', v', w) = \Pi(0, 0, w) = \Pi(0, v', 0) + \Pi(u'', 0, 0) + \text{alg. and logarith. function.}$$

At the end of this memoir are found certain theorems relative to the transformation of elliptic integrals (*cf.* Hermite, 'Cours à la Faculté des Sciences de Paris,' 4ème éd., 1891); from these theorems formulæ are deduced which set forth in a beautiful manner many problems of trans-

[1] See also Hermite ('Sur la division des fonctions Abéliennes,' *Mémoires des Savants étrangers*, 1848, p. 572); and Richelot (Crelle, bd. xxix. p. 281; and *Liouville's Journ.* 1843, p. 505).

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formation, multiplication, and division that are found in Jacobi's 'Fund. Nova.' Among others we may mention a direct method of the transformation of elliptic functions of the third kind, without presupposing, as is done by Jacobi, the formula of the transformation of functions of the second kind.

(41) Jacobi, in the eleventh section of his memoir 'De functionibus duarum variabilium quadrupliciter periodicis,' states without demonstration that if we have

$$x = \lambda(u, v),\ y = \lambda_1(u,v),$$

then the functions

$$x_n = \lambda(nu,nv),\ y_n = \lambda_1(nu,nv)$$

are by Abel's theorem obtained as the roots of a quadratic equation

$$\mathrm{U}x_n^2 + \mathrm{U}'x_n + \mathrm{U}'' = 0,$$

in which U, U', U'' are rational functions of x, y, $\sqrt{\mathrm{X}}, \sqrt{\mathrm{Y}}$ (see art. 37); and from this it is seen, *vice versâ*, that x and y may be obtained from x_n and y_n by the solution of algebraic equations.

Hermite (Crelle, bd. xxxii. pp. 176 and 277) finds that the coefficients U, U', and U'' are of the form $\mathrm{P} + \mathrm{Q}\,\Delta(x)\Delta y$, where P and Q are rational functions of x and y, and where

$$\Delta(x) = \sqrt{\mathrm{X}}\ (\text{above}) = \sqrt{x(1-x)(1-\kappa^2x)(1-\lambda^2x)(1-\mu^2x)}.$$

Let $i_1\sqrt{-1}$, i_2, $i_3\sqrt{-1}$, i_4 be the four periods of the integral $\int \frac{(\alpha+\beta x)}{\Delta(x)}dx$, and $i'_1\sqrt{-1}$, i'_2, $i'_3\sqrt{-1}$, i'_4 the corresponding periods of $\int \frac{(\alpha'+\beta' x)}{\Delta(x)}dx$; then the simultaneous roots of the two equations

$$(\mathrm{A.}) \qquad \mathrm{U}x_n^2+\mathrm{U}'x_n+\mathrm{U}''=0,\ \mathrm{U}y_n^2+\mathrm{U}'y_n+\mathrm{U}''=0$$

are expressed by Hermite through the formulæ

$$x=\lambda\left(u+\frac{mi_1\sqrt{-1}+m'i_2+m''i_3\sqrt{-1}+m'''i_4}{n},\right.$$
$$\left. v+\frac{mi'_1\sqrt{-1}+m'i'_2+m''i'_3\sqrt{-1}+m'''i'_4}{n}\right),$$
$$y=\lambda_1\left(u+\frac{mi_1\sqrt{-1}+m'i_2+m''i_3\sqrt{-1}+m'''i_4}{n},\right.$$
$$\left. v+\frac{mi'_1\sqrt{-1}+m'i'_2+m''i'_3\sqrt{-1}+m'''i'_4}{n}\right),$$

where m, m', m'', and m''' may take any of the values 0, 1, . . . $n-1$.

For brevity Hermite writes

$$\mathrm{I}=mi_1\sqrt{-1}+m'i_2+m''i_3\sqrt{-1}+m'''i_4,$$

and

$$\mathrm{I}'=mi'_1\sqrt{-1}+m'i'_2+m''i'_3\sqrt{-1}+m'''i'_4,$$

and by $f(x,y)$ is denoted any rational symmetric function of x and y, and p, q, r, s are used to indicate any four of the roots $x^n=1$.

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Hermite next proves that

$$(B.)\quad \sum_{m'''=0}^{m'''=n-1}\sum_{m''=0}^{m''=n-1}\sum_{m'=0}^{m'=n-1}\sum_{m=0}^{m=n-1} f\left[\lambda\left(u+\frac{\mathrm{I}}{n}, v+\frac{\mathrm{I}'}{n}\right), \lambda_1\left(u+\frac{\mathrm{I}}{n}, v+\frac{\mathrm{I}'}{n}\right)\right] p^m q^{m'} r^{m''} s^{m'''}$$

$$=\sqrt{\mathrm{A}+\mathrm{B}\Delta(\lambda(nu, nv))+\mathrm{C}\Delta(\lambda_1(nu, nv))+\mathrm{D}\Delta(\lambda(nu, nv))\ \Delta(\lambda_1(nu, nv))}$$

where A, B, C, D are rational functions of $\lambda(nu, nv)$ and $\lambda_1(nu, nv)$.

The first member of this equation may be denoted by $\phi(u, v)$, and may be expressed rationally in terms of $\lambda(u, v)$ and $\lambda_1(u, v)$, since, owing to the fundamental properties of the functions λ and λ_1, this is true of each of the terms constituting $\phi(u, v)$.

It is easily proved that

$$\phi\left(u+\frac{\kappa i_1\sqrt{-1}+\kappa' i_2+\kappa'' i_3\sqrt{-1}+\kappa''' i_4}{n}, v+\frac{\kappa i_1'\sqrt{-1}+\kappa' i'_2+\kappa'' i'_3\sqrt{-1}+\kappa''' i'_4}{n}\right)$$
$$=p^{-\kappa}q^{-\kappa'}r^{-\kappa''}s^{-\kappa'''}\phi(u, v),$$

whatever be the values of the integers κ, κ', κ'' and κ'''.

The nth power of ϕ is a rational function in $\lambda(u, v)$, $\lambda_1(u, v)$, which does not change when for these quantities are substituted any two other of the simultaneous roots of the proposed equations. It follows, therefore, from the theory of the symmetric functions of the roots of a system of equations in several unknown quantities, that this function may be rationally determined in the coefficients of the equation (A.); and since any rational function of two roots $\Delta(\lambda(nu, nv))$, $\Delta(\lambda_1(nu, nv'))$ may be put under the form

$$\mathrm{A}+\mathrm{B}\Delta(\lambda(nu, nv))+\mathrm{C}\Delta(\lambda_1(nu, nv))+\mathrm{D}\Delta(\lambda(nu, nv))\Delta(\lambda_1(nu, nv)),$$

the theorem is proved.

(42) Hermite says, in continuance of the above discussion, it seems, that the preceding considerations may be extended to the hyperelliptic integrals in general.

For let $\Delta(x)=\sqrt{x(1-x)(1-\lambda_1^2x)\ \ldots\ (1-\lambda^2{}_{2n+1}x)}$,

$$\theta_\alpha(x)=\alpha_\alpha+\beta_\alpha x+\gamma_\alpha x^2+\ \ldots\ +\eta_\alpha x^n,$$

$$\Phi_\alpha(x)=\int_0^x\frac{\theta_\alpha(x)dx}{\Delta(x)};$$

and write

$$u_i=\ldots_i(x_0)+\Phi_i(x_1)+\Phi_i(x_2)+\ \ldots\ +\Phi_i(x_n),$$
$$\text{and } x_i=\lambda_i(u_0, u_1, u_2, \ \ldots\ u_n),$$
$$(i=0, 1, 2, \ \ldots\ n).$$

Then $\Delta(x_i)=\theta_0(x_i)\frac{\partial x_i}{\partial u_0}+\theta_1(x_i)\frac{\partial x_i}{\partial u_1}+\theta_2(x_i)\frac{\partial x_i}{\partial u_2}+\ \ldots\ +\theta_n(x_i)\frac{\partial x_i}{\partial u_n}$,

$$(i=0, 1, 2, \ \ldots\ n),$$

where the partial derivatives may be rationally expressed in terms of the functions λ. Since θ is of the nth degree, it appears that the roots of the equation of the nth degree

$$0=\theta_0(x)\frac{\partial x_i}{\partial u_0}+\theta_1(x)\frac{\partial x_i}{\partial u_1}+\theta_2(x)\frac{\partial x_i}{\partial u_2}+\ \ldots\ +\theta_n(x)\frac{\partial x_i}{\partial u_n}$$

are the n functions $x_0, x_1, \ldots x_n$.

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In a letter to Liouville (*loc. cit.* p. 361), Hermite states that the representation of these functions is attended with great difficulties.

By supposing successively $f(x,y)=x+y$ and $f(x,y)=x.y$, the preceding theorem will give, expressed by a sum of nth roots, in number n^4-1, the coefficients of an equation of the second degree, whose roots will determine those of the proposed equations. These n^4-1 roots will be expressed rationally in terms of four of them.

Hermite next discusses the division of the periods.

In the second paper mentioned above, Hermite derives *first* certain theorems, from which he deduces, among others, Jacobi's formula for the algebraic expression of sin am (x) by sin am $\left(\frac{x}{M}\right)$, and Abel's fundamental properties of elliptic functions which relate to the addition of the arguments; other formulæ, which involve Jacobi's H and Θ functions are given.

Applications are then made to functions of two arguments and four periods. Hermite writes the integrals of the third kind in the form

$$\text{(I.)}\quad \int\left\{\left(\frac{\Delta(a)}{x-a}+\frac{\Delta(b)}{x-b}\right)\frac{2dx}{\Delta x}+\left(\frac{\Delta(a)}{y-a}+\frac{\Delta(b)}{y-b}\right)\frac{2dy}{\Delta y}\right\},$$

the integral being subjected to vanish, when $x=0$ and $y=0$. Δx represents the square root of the polynomial $p_1x+p_2x^2+p_3x^3+p_4x^4+p_5x^5$.

After

$$\begin{array}{ll} x=\lambda(u,v) & y=\lambda_1(u,v) \\ a=\lambda(\alpha,\beta) & b=\lambda_1(\alpha,\beta) \end{array}$$

are substituted in (I.), this integral is denoted by $\Pi(u,v,\alpha,\beta)$. When the variables u and v are introduced into the integrals of the second kind

$$\int\left(\frac{x^2dx}{\Delta(x)}+\frac{y^2dy}{\Delta(y)}\right) \text{ and } \int\left(\frac{x^3dx}{\Delta(x)}+\frac{y^3dy}{\Delta(y)}\right),$$

they are first denoted by $(u,v)_1$ and $(u,v)_2$ respectively.

Two new integrals are defined by the relations:

$$\mathrm{E}_1(u,v)=2p_4(u,v)_1+3p_5(u,v)_2 \text{ and } \mathrm{E}_2(u,v)=p_5(u,v)_1.$$

The following theorem is then derived

$$\Pi(u,v,\alpha,\beta)-\Pi(\alpha,\beta,u,v)=p_5(\alpha v-\beta u)+\alpha\mathrm{E}_1(u,v)+\beta\mathrm{E}_2(u,v)-u\mathrm{E}_1(\alpha,\beta)-v\mathrm{E}_2(\alpha,\beta),$$

a formula in which is seen the law of interchange of parameter and argument (art. 33).

Hermite further defines a function Φ by the relation

$$\Phi(u,v,\alpha,\beta)=\Pi(u,v,\alpha,\beta)+u\mathrm{Z}_1(\alpha,\beta)+v\mathrm{Z}_2(\alpha,\beta)-c(\alpha v-\beta u),$$

where Z_1 and Z_2 are certain functions of E_1 and E_2 respectively, and c is a constant.

Interesting addition-formulæ are derived for the two functions Π and Φ (*cf.* report made by Lamé and Liouville, 'Comp. Rend.' xvii. and 'Liouv. Journ.' t. viii. p. 502).

(43) *The introduction of the theta-function.*—The new functions of four simultaneous periods which Jacobi had discovered were received with great enthusiasm by mathematicians. The Academy of Sciences at Copenhagen wished to see presented the analogous functions that are connected with the integrals of all algebraic functions, to which Abel's

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theorem may be applied.[1] The representation of these functions, however, was not forthcoming, and later the solution of this problem was set as a prize question by the Berlin Academy of Sciences.

Abel had shown that the elliptic function $x=\text{sn}\,u$ could be represented as the quotient of infinite products. Jacobi, with the thought of representing an infinite product by means of a transcendental function, introduced into analysis the so-called *θ-function*, which represents such a product in the form of a power series. Investigating further this transcendent, he discovered its marvellous properties, and made use of it in his further researches in the elliptic functions. Jacobi[2] founded the whole theory of elliptic functions upon this new transcendent, which made these functions remarkably clear and simple, as well as their applications, for example, to rotatory motion, the swing of the pendulum, and innumerable problems of physics and mechanics; also by it the realms of geometry were essentially widened, and many abstract properties of the theory of numbers were revealed in a new light.

Hence it appears that the θ-function showed itself of paramount importance for the study of mathematics during the Jacobian epoch, and as a prototype it served for the future development of the function-theory and of all mathematics.

(44) The elliptic function $x=\text{sn}\,u$, as shown by Jacobi, may be expressed as the quotient of two θ-functions, where the θ-function may be written in the form

$$\theta(u)=\sum_{m=-\infty}^{m=\infty} e^{m^2\tau+2mu},$$

in which m takes all integral values from $-\infty$ to $+\infty$, u is the variable, and the constant τ is determined from the two periods of the integral

$$u=\int_0^x \frac{dx}{\sqrt{X}},$$

where X is of the fourth degree in x; or, as Jacobi says, τ determines the modulus of the elliptic integral.

(45) *θ-functions of two arguments.*—Goepel,[3] and in an independent manner Rosenhain,[4] generalised the simple θ-function of one variable and formed analogous transcendents, the θ-functions of two variables

$$\theta(u,v)=\sum_{-\infty m,n}^{+\infty} e^{a_1m^2+a_2mn+a_3n^2+2mu+2nv},$$

where here both m and n take all possible integral values from $-\infty$ to $+\infty$, u and v are the variables, and the constants a_1, a_2, and a_3 are determined from the four periods of the integrals

$$\int_0^x \frac{dx}{\sqrt{X}} \text{ and } \int_0^x \frac{xdx}{\sqrt{X}},$$

[1] Jacobi, *Werke*, bd. ii. p. 517.

[2] Jacobi, *Fund. Nova*, p. 45; also *Werke*, bd. i., p. 497. More recently Schellbach has made the θ-function his starting-point in his book, *Die Lehre von elliptischen Integralen und den Theta-Functionen*, Berlin, 1864.

[3] Goepel, *Theoriæ transcendentium Abelianarum primi ordinis adumbratio levis* (Crelle, bd. xxxv. p. 277, 1847).

[4] Rosenhain, *Mémoire sur les fonctions de deux variables à quatre périodes, &c.*, *Mém. des Savants étrangers*, t. xi. p. 361; see also Crelle, bd. xl. p. 319. Further see Jacobi, *Notiz über A. Goepel*, Crelle, bd. xxxv. p. 313.

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where X is a rational integral function of the fifth or sixth degree [1] in x. By means of the quotients of two such θ-functions Goepel and Rosenhain showed how to represent the functions $\phi(u,v)$ and $\psi(u,v)$ (art. 37), and thus completely solved the problem of representing the inverse functions of the hyperelliptic integrals of the first order.

The zeros and periodic properties of these two θ-functions, the relations between the squares of such functions and of the constants that enter these relations, the number of independent relations, Goepel's biquadratic relation, the connection between these functions and Kummer's sixteen-nodal quartic surface, and similar questions, are found in Harkness and Morley's 'A Treatise on the Theory of Functions,' p. 341 *et seq.*

(46) Goepel remarked that his investigations could be extended to any number of variables; but in this connection Jacobi showed that there is a troublesome paradox (Werke,' bd. ii. p. 521; and Weierstrass, 'Werke,' bd. i. p. 142); since, when there are more than two variables, the generalised θ-function contains more essential constants than the hyperelliptic functions with like number of variables.

(47) We must mention next papers by Hermite, 'Sur la théorie de la transformation des fonctions Abéliennes' ('Compt. Rend.' xl. pp. 249, 303, 365, 427, 485, 536, 704, and 784).

Besides the sum and the quotient of x and y (of art. 37), which we saw could be expressed through fractions whose numerator and denominator are functions of the argument u and v, and have unique and finite values for all finite real and imaginary values of these arguments, Goepel and Rosenhain gave in an analogous form the analytical expression of thirteen other functions of u and v, which depend algebraically but in an irrational manner upon the first two.

Hermite [2] designates by $f_1(u,v)$, $f_2(u,v)$, . . . $f_{15}(u,v)$ this complete system of fifteen functions which appear in the study of the integrals

$$\text{(I.)}\quad \begin{cases} \displaystyle\int_{x_0}^{x} \frac{dx}{\sqrt{\phi(x)}} + \int_{y_0}^{y} \frac{dx}{\sqrt{\phi(x)}} = u, \\ \displaystyle\int_{x_0}^{x} \frac{x\,dx}{\sqrt{\phi(x)}} + \int_{y_0}^{y} \frac{y\,dy}{\sqrt{\phi(y)}} = v, \end{cases}$$

when $\phi(x)$ denotes a polynomial of fifth or sixth degree in x, and which are analogous to the functions sn u, cn u, and dn u of the elliptic inte-

[1] Cayley ('Memoir on the Single and Double θ-function,' *Phil. Trans.* 1880, pp. 897–1002) treats the whole theory in a manner analogous to that employed by Goepel. In this paper special attention is paid to the relations among the squares of the functions and to the derivation of the biquadratic relation among four of the functions, which is the same as Kummer's sixteen-nodal quartic surface. See also Cayley (Crelle, bd. lxxxiii. pp. 210 and 235; and Forsyth (Biographical Notice on Arthur Cayley, 'Obituary Notices' of the *Proc. Royal Society*, vol. lviii.), and Cayley's *Math. Papers*, vol. viii. p. ix, where other references are given. Other papers on the same subject by Cayley are found in Crelle, bd. lxxxv. lxxxvii. and lxxxviii. Prof. Forsyth, 'Memoir on the Theta-function, particularly those of two Variables' (*Phil. Trans.* 1882, vol. clxxiii. p. 783) follows more closely Rosenhain's paper, and extends it in many directions. *Cf.* also Königsberger (Crelle, bd. lxiv. p. 17; bd. lxxxi. p. 193; and especially bd. lxxxvii. p. 173, where the problem of transformation is discussed fully), and also *Math. Ann.* bd. xv. p. 174.

[2] Hermite, *Sur la théorie de la transformation des fonctions Abéliennes* (*Compt. Rendus*, t. xl. pp. 249, 303, 365, 427, 485, 536, 704, 784).

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grals. In a similar manner he denotes by $F_1(u,v)$, $F_2(u,v)$, . . . $F_{15}(u,v)$ the functions of a similar nature to which one would come in taking for point of departure the equations

$$(\text{II.}\quad \begin{cases} \int_{x_0}^{x} \frac{\alpha+\beta x}{\sqrt{\psi(x)}}dx + \int_{y_0}^{y} \frac{\alpha+\beta y}{\sqrt{\psi(y)}}dy = u, \\ \int_{x_0}^{x} \frac{\gamma+\delta x}{\sqrt{\psi(x)}}dx + \int_{y_0}^{y} \frac{\gamma+\delta y}{\sqrt{\psi(y)}}dy = v, \end{cases}$$

where α, β, γ, and δ are constants, and $\psi(x)$ a polynomial of the fifth or sixth degree in x.

Hermite proposes as follows the problem of transformation: The polynomial $\psi(x)$ in (I.) being given, determine the coefficients of $\psi(x)$, and the constants α, β, γ, and δ in such a manner that the fifteen functions $F(u,v)$ be rationally expressed in terms of the fifteen functions $f(u,v)$.

By comparison of the linear relations that must exist among the periods of the $f(u,v)$ functions and the $F(u,v)$ functions, and of the relations that exist among the periods that belong to these respective functions, many remarkable consequences are deduced. In this connection see a letter of Eisenstein to Hermite ('Liouv. Journ.' xvii.) and also Eisenstein, 'Ueber die Vergleichung von solchen ternären quadratischen Formen, welche verschiedene Determinanten haben' ('Sitzungsber. der Berlin. Akad.' June 1852), and Hermite (Crelle, bd. xlvii. p. 343). From these papers is seen the intimate relation that exists between the analytic theory of transformation and the arithmetical theory of quadratic forms.

In the execution of the transformation a system of sixteen θ-functions is introduced, sixteen functions which may be algebraically expressed in terms of any two of them.

Four new functions, Π_0, Π_1, Π_2, and Π_3, are introduced, which may be expressed by an integral homogeneous function of degree κ in four of the θ-functions. The Π-functions contain linearly $\frac{\kappa^2+1}{2}$ constants. There are just enough conditions of the problem to determine these constants. Further, the Π-functions are defined in terms of other Θ-functions. From this follow immediately relations among the quadruply-periodic quotients which arise from the division of two Θ-functions and those which arise from the division of two θ-functions. These last functions may be regarded as representing the more general periodic functions which orginate from the hyperelliptic integrals of the first order, when the arguments x and y have been replaced by others which depend linearly upon them in any manner.

Thus the proposed problem of transformation is solved.[1] On p. 704 of the 'Compt. Rendus,' t. xl., Hermite gives a method of division of the θ-functions, and compares them with Weierstrass' Al-functions of the following article.

Liouville and Hermite made use of the periodic properties of the single θ-functions, and derived for the elliptic functions the results of addition, multiplication, transformation, and division; and Hermite by direct trans-

[1] *Cf.* Brioschi (*Compt. Rend.* xlvii. p. 310).

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formations gained a clearer insight into the properties of the θ-functions of two variables. See Liouville ('Compt. Rendus,' 1851); and Hermite, (Crelle, bd. xxxii. pp. 176 and 277).

(48) As those mathematicians whose works were mentioned in arts. 1–11 laid the foundation for the investigations of Jacobi and Abel, so may we also regard the works that have been reviewed up to the present time as introductory to the works of Weierstrass and Riemann.

The theory of Abelian functions has been so generalised, so widened, by these mathematicians and their followers that we may make the same remark concerning it as Jacobi (in a letter to Crelle, 'Crelle's Journ.' bd. iii. p. 310) made regarding the elliptic functions: 'You see that the theory is a vast subject of research, which in the course of its development embraces almost all algebra, the theory of definite integrals and the science of numbers.'

As suggested in the introduction, an account of the works of Weierstrass, Riemann, Clebsch, and later writers cannot be given in this report owing to the lack of space required for such a statement. To leave the work thus unfinished would cause the author much regret; however, there has just appeared the admirable treatise of Mr. H. F. Baker on 'Abel's Theorem and the Allied Theory,' in which the discoveries of the mathematicians just mentioned and the development of the theory of Abelian functions are treated in a very comprehensive and elegant manner.

In conclusion, the author takes much pleasure in referring to this book, at the beginning of which we are taught that no better guide can be found to the analytical developments of pure mathematics than a study of the theory of Abelian functions.

The Action of Light upon Dyed Colours.—Report of the Committee, consisting of Professor T. E. THORPE (*Chairman*), Professor J. J. HUMMEL (*Secretary*), Dr. W. N. PERKIN, Professor W. J. RUSSELL, Captain ABNEY, Professor W. STROUD, *and* Professor R. MELDOLA. (*Drawn up by the Secretary.*)

DURING the past year (1896-97) the work of this Committee has been continued as usual, and a large number of wool and silk patterns, dyed with various natural and artificial *brown* and *black* colouring matters, have been examined with respect to their power of resisting the fading action of light.

The Committee regret to state, however, that at this meeting of the Association they are unable to give an account of the results obtained, since at the earlier date this year at which Reports of Committees had to be sent to the Organising Committees the dyed patterns were still under exposure to light, so that the Report could not be prepared. It will be presented, however, at the next meeting of the Association. The Committee ask for reappointment, and for a grant of 8*l.* to carry on the work.

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